Gravitational Lighting Bending: A Quantum Field Theory Perspective

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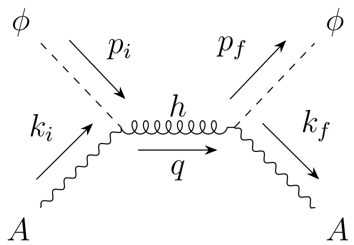
1 Introduction

We want to derive the classic result of light bending around a sufficiently massive object from purely QFT considerations. It's a relatively simple exercise, but it shows that many results of general relativity necessarily follows from the mediator of gravitational interactions being a massless spin-2 particle. This document is essentially filling in some of the details that can be found in this paper arXiv:1704.05067.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}\,\text{GeV}$. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.



2 Light Bending

Any interaction between fields starts from the action. The set up as a massive object with mass M whose gravitational field bends an incoming null ray from its original path. In the original solar system test of GR, the culprit that was responsible for this was a star. Stars don't tend to rotate quickly which means their angular momentum tends to be pretty small. We will thus model stars as just a scalar field ϕ. Light, of course, we be modeled with the usual massless spin-1 vector field A_μ. The action for this setup is given by

$$S = \int \mathrm{d}^4x \sqrt{-g}\left[\frac{2}{\kappa^2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 - \frac{1}{4}g^{\mu\nu}g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho}\right], \tag{1}$$

where $\kappa^2 = 32\pi G$ and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

Since light bending is a fairly low energy event, we need only to work in the weak-field limit. This means the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{3}$$

which brings the inverse metric and volume element to the forms

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \ldots, \qquad \sqrt{-g} = 1 + \frac{\kappa}{2}h, \tag{4}$$

with $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. Plugging this all in while only including at most one power of the metric perturbation or equivalently one power of κ leads us with

$$S = \int \mathrm{d}^4x \left(1 + \frac{\kappa}{2}h\right)\left[\frac{2}{\kappa^2}R^{(2)} - \frac{1}{2}(\eta^{\mu\nu} - \kappa h_{\mu\nu})\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}M^2\phi^2 - \frac{1}{4}(\eta^{\mu\nu} - \kappa h^{\mu\nu})(\eta^{\lambda\rho} - \kappa h^{\lambda\rho})F_{\mu\lambda}F_{\nu\rho}\right]. \tag{5}$$

After a bit of algebra, we're left with

$$S = \int \mathrm{d}^4x \left[\frac{2}{\kappa^2}R^{(2)} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 + \frac{\kappa}{2}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \right.$$
$$\left. -\frac{1}{4}(\eta^{\mu\nu}\eta^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho} - \kappa\eta^{\mu\nu}h^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho} - \kappa\eta^{\lambda\rho}h^{\mu\nu}F_{\mu\lambda}F_{\nu\rho})\right] \tag{6}$$
$$+ \int \mathrm{d}^4x\, \frac{\kappa}{2}h\Big(-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{1}{4}\eta^{\mu\nu}\eta^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho}\Big),$$

where $(\partial\phi)^2 \equiv \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. We can separate these two actions into a free action S_0

$$S_0 = \int \mathrm{d}^4x \left[\frac{2}{\kappa^2}R^{(2)} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}M^2\phi^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right], \tag{7}$$

i.e. the action that yields the equations of motion and the interaction Lagrangians. First we symmetrize the coupling between the kinetic energy of the scalar field and the metric perturbation

$$h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{1}{2}(h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + h^{\nu\mu}\partial_\nu\phi\partial_\mu\phi) = \frac{1}{2}h^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi + \partial_\nu\phi\partial_\mu\phi), \tag{8}$$

which gives the graviton-scalar interaction Lagrangian as being

$$\mathcal{L}_{h\phi\phi} = \frac{\kappa}{4}h^{\mu\nu}\big(\partial_\mu\phi\partial_\nu\phi + \partial_\nu\phi\partial_\mu\phi - \eta_{\mu\nu}[(\partial\phi)^2 + M^2\phi^2]\big), \tag{9}$$

and the gravity-vector interaction Lagrangian is given by

$$\mathcal{L}_{hAA} = -\frac{\kappa}{4}\left(\eta^{\mu\nu}h^{\lambda\rho} + \eta^{\lambda\rho}h^{\mu\nu} + \frac{1}{2}h\eta^{\mu\nu}\eta^{\lambda\rho}\right)F_{\mu\lambda}F_{\nu\rho}. \tag{10}$$

Now we want to derive the Feynman/vertex rules for each interaction. The easiest way to do this is by taking a page from the path integral formalism i.e. taking repeated derivatives and replace derivatives with momenta i.e. $\partial_\mu \to -ip_\mu$. As the Feynman diagram indicates, we use $p_{i,f}, k_{i,f}$ for the initial/final momentum for the scalar and vector respectively. We also have the initial momenta to be incoming and the final momenta to be out going. Taking the first functional derivative gives

$$\frac{\delta\mathcal{L}_{h\phi\phi}}{\delta\phi} = \frac{\kappa}{4}h^{\mu\nu}\big(2(-ip_\mu)\partial_\nu\phi + 2(-ip_\nu)\partial_\mu\phi - \eta_{\mu\nu}[2(-ip^\lambda)\partial_\lambda\phi + 2M^2\phi]\big). \tag{11}$$

Taking one more derivative gives

$$\frac{\delta^2\mathcal{L}_{h\phi\phi}}{\delta\phi^2} = \frac{\kappa}{2}h_{\mu\nu}\big(p_i^\mu p_f^\nu + p_f^\mu p_i^\nu - \eta^{\mu\nu}[p_i\cdot p_f + M^2]\big), \tag{12}$$

where for outgoing momenta, we replaced the derivatives by $\partial_\mu \to ip_\mu$. Thus, the vertex rule for scalar-graviton interactions is

$$V^{\mu\nu}(p_i,p_f) = \frac{-i\kappa}{2}\big(p_i^\mu p_f^\nu + p_f^\mu p_i^\nu - \eta^{\mu\nu}[p_i\cdot p_f + M^2]\big). \tag{13}$$

Now we do the same for the vector-graviton interaction. The only difference is that in addition to replacing the derivatives, we must also replace the vector field A^μ with polarization vectors i.e. $A^\mu \to \epsilon_\lambda^\mu$ where λ is the polarization of the photon. Writing

$$F_{\mu\lambda}F_{\nu\rho} = \partial_\mu A_\lambda \partial_\nu A_\rho - \partial_\mu A_\lambda \partial_\rho A_\nu - \partial_\lambda A_\mu \partial_\nu A_\rho + \partial_\lambda A_\mu \partial_\rho A_\nu, \tag{14}$$

the first derivative/replacement leaves us with

$$\begin{aligned}\frac{\delta\mathcal{L}_{hAA}}{\delta A} &= -\frac{\kappa}{4}\left(\eta_{\mu\nu}h_{\lambda\rho} + \eta_{\lambda\rho}h_{\mu\nu} + \frac{1}{2}h\eta_{\mu\nu}\eta_{\lambda\rho}\right)\times \\ &\times [(-ik_i^\mu\epsilon_i^\lambda)(\partial^\nu A^\rho) + \partial^\mu A^\lambda(-ik_i^\nu\epsilon_i^\rho) - (-ik_i^\mu\epsilon_i^\lambda)\partial^\rho A^\nu - \partial^\mu A^\lambda(-ik_i^\rho\epsilon_i^\lambda) \\ &- (-ik_i^\lambda\epsilon_i^\mu)\partial^\nu A^\rho - (-ik_i^\nu\epsilon_i^\rho)\partial^\lambda A^\mu + (-ik_i^\lambda\epsilon_i^\mu)\partial^\rho A^\nu - \partial^\lambda A^\mu(-ik_i^\rho\epsilon_i^\nu)],\end{aligned} \tag{15}$$

here i represent the initial polarization of the photon i.e. $\epsilon_i \equiv \epsilon(k_i)$. Next we take the derivative again we get

$$\begin{aligned}\frac{\delta^2\mathcal{L}_{hAA}}{\delta A^2} &= -\frac{\kappa}{4}\left(\eta_{\mu\nu}h_{\lambda\rho} + \eta_{\lambda\rho}h_{\mu\nu} + \frac{1}{2}h\eta_{\mu\nu}\eta_{\lambda\rho}\right)\times \\ &\times [(-ik_i^\mu\epsilon_i^\lambda)(ik_f^\nu\epsilon_f^\rho) + (ik_f^\mu\epsilon_f^\lambda)(-ik_i^\nu\epsilon_i^\rho) - (-ik_i^\mu\epsilon_i^\lambda)(ik_f^\rho\epsilon_f^\nu) - (ik_f^\mu\epsilon_f^\lambda)(-ik_i^\rho\epsilon_i^\lambda) \\ &- (-ik_i^\lambda\epsilon_i^\mu)(ik_f^\nu\epsilon_f^\rho) - (-ik_i^\nu\epsilon_i^\rho)(ik_f^\lambda\epsilon_f^\mu) + (-ik_i^\lambda\epsilon_i^\mu)(ik_f^\rho\epsilon_f^\nu) - (ik_f^\lambda\epsilon_f^\mu)(-ik_i^\rho\epsilon_i^\nu)],\end{aligned} \tag{16}$$

and after some algebra results in

$$\begin{aligned}\frac{\delta^2\mathcal{L}_{hAA}}{\delta A^2} &= -\frac{\kappa}{2}\{h_{\mu\nu}[(k_i\cdot k_f)(\epsilon_i^\mu\epsilon_f^\nu + \epsilon_i^\nu\epsilon_f^\mu) + (\epsilon_i\cdot\epsilon_f)(k_i^\mu k_f^\nu + k_i^\nu k_f^\mu) \\ &- (k_i\cdot\epsilon_f)(\epsilon_i^\mu k_f^\nu + \epsilon_i^\nu k_f^\mu) - (\epsilon_i\cdot k_f)(k_i^\mu\epsilon_f^\nu + k_i^\nu\epsilon_f^\mu)] \\ &- h[(k_i\cdot k_f)(\epsilon_i\cdot\epsilon_f) - (k_i\cdot\epsilon_f)(k_f\cdot\epsilon_i)]\}.\end{aligned} \tag{17}$$

Now this result is fine, but its not particularly helpful as a vertex rule. We would like to know what the graviton-vector vertex rule is so it'll be easier in the future to take inner products of the various components. Let's try to factor out the metric perturbation and polarization vectors $\epsilon_i^\lambda, \epsilon_f^\rho$ from the above expression. Since $h_{\mu\nu}$ is the easier of the two, we can start from that

$$\begin{aligned}\frac{\delta^2\mathcal{L}_{hAA}}{\delta A^2} =& -\frac{\kappa}{2}h_{\mu\nu}\{[(k_i\cdot k_f)(\epsilon_i^\mu\epsilon_f^\nu+\epsilon_i^\nu\epsilon_f^\mu)+(\epsilon_i\cdot\epsilon_f)(k_i^\mu k_f^\nu+k_i^\nu k_f^\mu)\\ &-(k_i\cdot\epsilon_f)(\epsilon_i^\mu k_f^\nu+\epsilon_i^\nu k_f^\mu)-(\epsilon_i\cdot k_f)(k_i^\mu\epsilon_f^\nu+k_i^\nu\epsilon_f^\mu)]\\ &-\eta_{\mu\nu}[(k_i\cdot k_f)(\epsilon_i\cdot\epsilon_f)-(k_i\cdot\epsilon_f)(k_f\cdot\epsilon_i)]\},\end{aligned} \tag{18}$$

which leads us to rearrange the terms to get

$$\begin{aligned}\frac{\delta^2\mathcal{L}_{hAA}}{\delta A^2} =& -\frac{\kappa}{2}h_{\mu\nu}[(k_i\cdot k_f)[\epsilon_i^\mu\epsilon_f^\nu+\epsilon_i^\nu\epsilon_f^\mu-\eta^{\mu\nu}(\epsilon_i\cdot\epsilon_f)]+(\epsilon_i\cdot\epsilon_f)(k_i^\mu k_f^\nu+k_i^\nu f_f^\mu)\\ &+\eta^{\mu\nu}(k_i\cdot\epsilon_f)(k_f\cdot\epsilon_i)-(k_i\cdot\epsilon_f)(\epsilon_i^\mu k_f^\nu+\epsilon_i^\nu k_f^\mu)-(\epsilon_i\cdot k_f)(k_i^\mu\epsilon_f^\nu+k_i^\nu\epsilon_f^\mu)].\end{aligned} \tag{19}$$

This can be further expressed by writing $\epsilon_i\cdot\epsilon_f=\eta_{\lambda\rho}\epsilon_i^\lambda\epsilon_f^\rho$ as well as $\epsilon_i^\mu=\eta_\lambda^\mu\epsilon_i^\lambda$ and $k_i\cdot\epsilon_f=\eta_{\alpha\rho}k_i^\alpha\epsilon_f^\rho$ to get

$$\begin{aligned}\frac{\delta^2\mathcal{L}_{hAA}}{\delta A^2} =& -\frac{\kappa}{2}h_{\mu\nu}[k_i\cdot k_f(\eta_\lambda^\mu\epsilon_i^\lambda\eta_\rho^\nu\eta_f^\rho+\eta_\lambda^\nu\epsilon_i^\lambda\eta_\rho^\mu\eta_f^\rho-\eta^{\mu\nu}\eta_{\lambda\rho}\epsilon_i^\lambda\epsilon_f^\rho)+(k_i^\mu k_f^\nu+k_i^\nu f_f^\mu)\eta_{\lambda\rho}\epsilon_i^\lambda\epsilon_f^\rho\\ &-\eta_{\alpha\rho}k_i^\alpha\epsilon_f^\rho(\eta_\lambda^\mu\epsilon_i^\lambda k_f^\nu+\eta_\lambda^\nu\epsilon_i^\lambda k_f^\mu)-\eta_{\alpha\lambda}\epsilon_i^\lambda k_f^\alpha(k_i^\mu\eta_\rho^\nu\epsilon_f^\rho+k_i^\nu\eta_\rho^\mu\epsilon_f^\rho)+\eta^{\mu\nu}\eta_{\alpha\rho}k_i^\alpha\epsilon_f^\rho\eta_{\beta\lambda}k_f^\beta\epsilon_i^\lambda].\end{aligned} \tag{20}$$

This empowers us to write

$$\begin{aligned}\frac{\delta^2\mathcal{L}_{hAA}}{\delta A^2} =& -\frac{\kappa}{2}h_{\mu\nu}\epsilon_i^\lambda\epsilon_f^\rho[k_i\cdot k_f(\eta_\lambda^\mu\eta_\rho^\nu+\eta_\rho^\mu\eta_\lambda^\nu-\eta^{\mu\nu}\eta_{\lambda\rho})+\eta_{\lambda\rho}(k_i^\mu k_f^\nu+k_i^\nu k_f^\mu)-\eta_{\alpha\rho}(\eta_\lambda^\mu k_f^\nu+\eta_\lambda^\nu k_f^\mu)\\ &+\eta^{\mu\nu}\eta_{\alpha\rho}\eta_{\beta\lambda}k_i^\alpha k_f^\beta-\eta_{\alpha\lambda}(k_i^\mu\eta_\rho^\nu+k_i^\nu\eta_\rho^\mu)]\\ =& -\frac{\kappa}{2}h_{\mu\nu}\epsilon_i^\lambda\epsilon_f^\rho[k_i\cdot k_f(\eta_\lambda^\mu\eta_\rho^\nu+\eta_\rho^\mu\eta_\lambda^\nu-\eta^{\mu\nu}\eta_{\lambda\rho})+\eta_{\lambda\rho}(k_i^\mu k_f^\nu+k_i^\nu k_f^\mu)\\ &-k_{i,\rho}(\eta_\lambda^\mu k_f^\nu+\eta_\lambda^\nu k_f^\mu)+\eta^{\mu\nu}k_{i,\rho}k_{f,\lambda}-k_{f,\lambda}(k_i^\mu\eta_\rho^\nu+k_i^\nu\eta_\rho^\mu)].\end{aligned} \tag{21}$$

Thus we can now read off what the vertex rule is

$$\begin{aligned}V^{\mu\nu\lambda\rho}(k_i,k_f) =& \frac{i\kappa}{2}[k_i\cdot k_f(\eta^{\mu\lambda}\eta^{\nu\rho}-\eta^{\mu\nu}\eta^{\lambda\rho})+\eta^{\mu\nu}k_i^\rho k_f^\lambda+\eta^{\lambda\rho}(k_i^\mu k_f^\nu+k_i^\nu k_f^\mu)\\ &-(\eta^{\mu\lambda}k_i^\rho k_f^\nu+\eta^{\mu\rho}k_i^\nu k_f^\lambda+\eta^{\nu\lambda}k_i^\rho k_f^\mu+\eta^{\nu\rho}k_i^\mu k_f^\lambda)].\end{aligned} \tag{22}$$

Our last main ingredient as the propagator for a massless spin-2 particle. The propagator is given by

$$P_{\mu\nu\alpha\beta}(q) = \frac{1}{2q^2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}). \tag{23}$$

We are finally in a position to calculate the total amplitude. There are two contributions: one from the diagram we listed above and the other where we cross the legs from the photons. Thus our amplitudes are

$$i\mathcal{M}^I = V^{\mu\nu}(p_i, p_f)P_{\mu\nu\alpha\beta}(q)V^{\alpha\beta}{}_{\lambda\rho}(k_i, k_f)\epsilon_i^\lambda(k_i)\epsilon_f^\rho(k_f), \tag{24}$$

and

$$i\mathcal{M}^{II} = V^{\mu\nu}(p_i, p_f)P_{\mu\nu\alpha\beta}(q)V^{\alpha\beta}{}_{\lambda\rho}(k_i, k_f)\epsilon_i^\lambda(k_f)\epsilon_f^\rho(k_i). \tag{25}$$

Now we want to make a few approximations. First, we're working in the static limit so therefore

$$(p_i)_\mu \approx (p_f)_\mu \approx M\eta_{\mu 0} = (-M, 0, 0, 0), \tag{26}$$

which means the massive object/star's momentum is almost completely unaltered by the scattering light. We also label

$$k_i \equiv (\omega_i, \mathbf{k}_i), \qquad k_f \equiv (\omega_f, \mathbf{k}_f). \tag{27}$$

We also assume that the energy/momentum transfer will be very small as well i.e.

$$(k_i - k_f)^2 = -2k_i \cdot k_f = -2[-\omega_i\omega_f + \mathbf{k}_i \cdot \mathbf{k}_f] = 2\omega_i\omega_f(1-\cos\theta) = 4\omega_i\omega_f \sin^2\left(\frac{\theta}{2}\right) \approx 4\omega^2 \sin^2\left(\frac{\theta}{2}\right), \tag{28}$$

where we used the fact that $k_i^2 = k_f^2 = 0$ and thus $\mathbf{k}_{i,f} = \omega\hat{\mathbf{k}}_{i,f}$ with the energy of the photon is basically unchanged[1] as well so $\omega_i \approx \omega_f$. Noting $p_i \cdot p_f = -M^2$, we have

$$V^{\mu\nu}(p_i, p_f) = \frac{-i\kappa}{2}(p_i^\mu p_f^\nu + p_i^\nu p_f^\mu - \eta^{\mu\nu}(p_i \cdot p_f + M^2)) \approx -i\kappa M^2 \eta_0^\mu \eta_0^\nu, \tag{29}$$

[1] This follows from the static approximation of the star that we're using.

which when contracted with the graviton-propagator yields

$$V^{\mu\nu}(p_i,p_f)P_{\mu\nu\alpha\beta}(q) = -i\kappa M^2 P_{00\alpha\beta} = \frac{-i\kappa M^2}{2q^2}(2\eta_{\alpha 0}\eta_{\beta 0} + \eta_{\alpha\beta}). \tag{30}$$

Now let's contract the above tensor with the graviton-vector vertex rule

$$\begin{aligned} V^{\mu\nu}P_{\mu\nu\alpha\beta}V^{\alpha\beta\lambda\rho} =& \frac{-i\kappa M^2}{2q^2}(2\eta_{\alpha 0}\eta_{\beta 0} + \eta_{\alpha\beta})\frac{i\kappa}{2}[k_i\cdot k_f(\eta^{\alpha\lambda}\eta^{\beta\rho} - \eta^{\alpha\beta}\eta^{\lambda\rho}) + \eta^{\alpha\beta}k_i^\rho k_f^\lambda \\ &+ \eta^{\lambda\rho}(k_i^\alpha k_f^\beta + k_i^\beta k_f^\alpha) - (\eta^{\alpha\lambda}k_i^\rho k_f^\beta + \eta^{\alpha\rho}k_i^\beta k_f^\lambda + \eta^{\beta\lambda}k_i^\rho k_f^\alpha + \eta^{\beta\rho}k_i^\alpha k_f^\lambda)] \\ =& \frac{\kappa^2 M^2}{4q^2}[4k_i\cdot k_f\eta_0^\lambda\eta_0^\rho + 2k_i\cdot k_f\eta^{\lambda\rho} - 2k_i^\rho k_f^\lambda + 4k_{i,0}k_{f,0}\eta^{\lambda\rho} \\ &- 2\eta_0^\lambda k_i^\rho k_{f,0} - 2\eta_0^\rho k_{i,0}k_f^\lambda - 2\eta_0^\lambda k_i^\rho k_{f,0} - 2\eta_0^\rho k_{i,0}k_f^\lambda]. \end{aligned} \tag{31}$$

Using the condition that for a massless vector, $\epsilon^\mu = (0,\hat{\epsilon})$, contracting the above term with the polarization vector gives

$$V^{\mu\nu}P_{\mu\nu\alpha\beta}V^{\alpha\beta}{}_{\lambda\rho}\,\epsilon_i^\lambda\epsilon_f^\rho = \frac{\kappa^2 M^2}{4q^2}\big[2k_i\cdot k_f(\epsilon_i\cdot\epsilon_f) - 2(\epsilon_i\cdot k_f)(\epsilon_f\cdot k_i) + 4\omega^2(\epsilon_i\cdot\epsilon_f)\big]. \tag{32}$$

Lastly, we'll assume that the deviation in the angle between the incoming and outgoing photon will be small and thus[2] $k_i\cdot k_f \approx k_i\cdot\epsilon_f \approx \epsilon_i\cdot k_f \approx 0$. Recognizing that conservation of energy/momentum at the vertices means $q^2 = (k_i - k_f)^2$, leaves us with the following amplitude

$$i\mathcal{M}^I \approx \frac{\kappa^2 M^2}{4(4\omega^2\sin^2\left(\frac{\theta}{2}\right))}(4\omega^2\epsilon_i\cdot\epsilon_f) = \frac{\kappa^2 M^2}{4\sin^2\left(\frac{\theta}{2}\right)}\epsilon_+(k_i)\cdot\epsilon_-(k_f), \tag{33}$$

where we use "+" and "-" to denote the different polarizations of the photon. The second amplitude will be virtually identical to the first one, all we need to do is swap the polarizations/momenta of the vectors

$$i\mathcal{M}^{II} \approx \frac{\kappa^2 M^2}{4\sin^2\left(\frac{\theta}{2}\right)}\epsilon_-(k_i)\cdot\epsilon_+(k_f). \tag{34}$$

[2]The condition $k_i\cdot\epsilon_f \approx 0$ comes from the approximation that since the initial and final direction of the photon doesn't change very much, the propagating direction of the final photon will be pretty close to the initial photon and thus the direction of the initial photon momentum will be pretty close to being transverse to the direction of the outgoing photon. Then its just a matter of $k_i\cdot\epsilon_f \approx k_i\cdot\epsilon_i = 0$ and likewise if we were to swap the i and f indices.

We are ultimately interested in calculating the cross section in the center of mass frame which is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(8\pi)^2 s}|\mathcal{M}|^2, \tag{35}$$

where $s = -(p_i + k_i)^2$ is the Mandelstam variable which gives the total energy in the center of mass frame. In our case, this can be expressed as

$$s = -(p_i + k_i)^2 = -p_i^2 - 2p_i \cdot k_i = M^2 - 2M\omega \approx M^2, \tag{36}$$

since we're taking the energy of the photon to be small too. For a photon that initially propagates in the z-direction, a common choice for the polarization vectors are given by

$$\epsilon_{\pm}(k) = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0), \tag{37}$$

and therefore the dot product yields

$$\epsilon_+(k_i) \cdot \epsilon_-(k_f) = \epsilon_-(k_i) \cdot \epsilon_+(k_f) = \frac{1}{2}(-1-1) = -1. \tag{38}$$

Averaging over initial and summing over final photon polarizations then gives us

$$\frac{1}{2}\sum_{h=\pm} |\epsilon_h(k_i) \cdot \epsilon_{-h}(k_f)|^2 = \frac{1}{2}(|-1|^2 + |-1|^2) = 1. \tag{39}$$

Thus, the differential cross section can be simply written as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2(8\pi)^2 s}\left|\frac{\kappa^2 M^2}{4\sin^2\left(\frac{\theta}{2}\right)}\right|^2 \sum_{h=\pm} |\epsilon_h(k_i) \cdot \epsilon_{-h}(k_f)|^2. \tag{40}$$

Using the small-angle approximation of $\sin\left(\frac{\theta}{2}\right) \approx \theta/2$, the differential cross section becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \frac{1}{2 \cdot 64\pi^2 M^2} \times \frac{(32\pi G)^2 M^4}{16(\theta/2)^4} \times 2 = \frac{16 G^2 M^2}{\theta^4}. \tag{41}$$

We're almost done! We have the differential cross section in terms of the angle and mass. The light-bending result is given in terms of the impact parameter. The cross section can be expressed in terms of the impact parameter by

$$\sigma = \pi b^2, \tag{42}$$

where we assume that the scattering is taking place within a disk of radius b. This implies

$$\mathrm{d}\sigma = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = 2\pi b\,\mathrm{d}b. \tag{43}$$

Recall that the solid angle measure is

$$\mathrm{d}\Omega = 2\pi \sin\theta\,\mathrm{d}\theta. \tag{44}$$

This implies that the differential cross section is related to the impact parameter by taking the ratio of the above two equations

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right|, \tag{45}$$

the absolute value sign is put in by hand to ensure the result is positive. Equating the expression we found for the differential cross section gives us

$$b\,\mathrm{d}b = -\frac{16G^2M^2}{\theta^4}\sin\theta\,\mathrm{d}\theta \approx -\frac{16G^2M^2}{\theta^3}\,\mathrm{d}\theta, \tag{46}$$

the minus sign is put in because the impact parameter is decreasing and so therefore its derivative is negative. Integrating both sides gives

$$\frac{b^2}{2} \approx \frac{8G^2M^2}{\theta^2}, \tag{47}$$

where we've ignored constants of integration. Solving for the deflection angle gives

$$\theta = \frac{4GM}{b}, \tag{48}$$

which is the famous light bending result of gravitational lensing!