The Einstein-Hilbert Action

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We want to vary the action with respect to the inverse metric in order to derive Einstein's (covariant) equations. This document will serve as a hub for looking up how to derive the equations of motion for various modified gravity theories. First we start with the classical Einstein-Hilbert action.

1 Einstein-Hilbert Equations of Motion

The Lagrangian takes the form

$$\mathcal{L} = \sqrt{-g}R, \tag{1}$$

where $g = \det(g_{\mu\nu})$ is the determinate of the metric tensor and R is the Ricci scalar. Under the action, we get the expression

$$S = \int \mathrm{d}^4x\,\mathcal{L}, \tag{2}$$

where we integrate over 4-dimensional spacetime. If one wishes to consider matter within the theory, all that is needed is the addition of the matter Lagrangian $\mathcal{L}_m$. The action becomes

$$S = \int \mathrm{d}^4x\,[\mathcal{L} + \mathcal{L}_m]. \tag{3}$$

From classical mechanics (CM), we are reminded that if one were interested in deriving the equations of motion for the physical system, we must vary with respect to the dynamical variable. In the case of CM, we varied with respect to time. Here though, since we're treating all of the spacetime coordinates as the same, that won't be sufficient. In GR, it is much more common to vary with respect to the inverse metric instead. First we write the above expression as

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \left[\sqrt{-g}R(g_{\mu\nu}) + \mathcal{L}_m\right]. \tag{4}$$

where $2\kappa^2 = 16\pi G$ is the Planck mass and G is Newton's constant. We will soon vary with respect to the inverse metric. But first we recognize that $R = g^{\mu\nu}R_{\mu\nu}$. So we then have

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \left[\sqrt{-g}(g^{\mu\nu}R_{\mu\nu} + \mathcal{L}_m)\right]. \tag{5}$$

Now we shall vary the action with respect to the inverse metric

$$\delta S = \frac{1}{2\kappa^2} \int \mathrm{d}^4x \left[(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})\sqrt{-g} + R\delta\sqrt{-g}\right] + \delta S_m \tag{6}$$

and S_m is the action for matter. First we want to deal with the variation in the volume. Recall we can treat the variation operator as a differential operator. We get

$$\delta\sqrt{-g} = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g, \tag{7}$$

where δg is the variation in the determinant. To figure out what this is, we want to look at how the determinant is defined

$$g = \sum_{\mu\nu} (-1)^{\mu+\nu} M^{\mu\nu} g_{\mu\nu}, \tag{8}$$

where $M^{\mu\nu}$ is the determinant of the determinant whose μ-row and $\nu-$column has been deleted. Thus, the variant in the determinant is simply

$$\delta g = (-1)^{\mu+\nu} M^{\mu\nu} \delta g_{\mu\nu}. \tag{9}$$

Next, we write the co-factor matrix in terms of the inverse metric and the determinant

$$g^{\mu\nu} = \frac{1}{g}(-1)^{\mu+\nu} M^{\mu\nu}, \tag{10}$$

and so the variant in the determinant becomes

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}. \tag{11}$$

Recall we wish to vary the action with respect to the *inverse* metric. To get the above expression in those terms, we use the fact that

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu} \Rightarrow g_{\lambda\nu} \delta g^{\mu\lambda} + g^{\mu\lambda} \delta g_{\lambda\nu} = 0. \tag{12}$$

The last expression implies that the variation in either the metric or inverse metric can be written as

$$\delta g_{\mu\nu} = -g_{\mu\lambda} g_{\nu\rho} \delta g^{\lambda\rho}, \quad \delta g^{\lambda\rho} = -g^{\lambda\mu} g^{\rho\nu} \delta g_{\mu\nu}. \tag{13}$$

So the variation in the volume form is merely

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}(-g g_{\mu\nu} \delta g^{\mu\nu}) = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \tag{14}$$

Now we are interested in figuring out the variation in the curvature tensor. To figure out what $\delta R_{\mu\nu}$ is, we will first look at the Riemann tensor and its variation

$$R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\lambda} - \partial_\nu \Gamma^{\rho}_{\mu\lambda} + \Gamma^{\rho}_{\mu\alpha} \Gamma^{\alpha}_{\nu\lambda} - \Gamma^{\rho}_{\nu\alpha} \Gamma^{\alpha}_{\mu\lambda}. \tag{15}$$

The variation in the Riemann tensor is then

$$\delta R^{\rho}{}_{\lambda\mu\nu} = \partial_\mu(\delta\Gamma^{\rho}_{\nu\lambda}) - \partial_\nu(\delta\Gamma^{\rho}_{\mu\lambda}) + (\delta\Gamma^{\rho}_{\mu\alpha})\Gamma^{\alpha}_{\nu\lambda} + \Gamma^{\rho}_{\mu\alpha}(\delta\Gamma^{\alpha}_{\nu\lambda}) - (\delta\Gamma^{\rho}_{\nu\alpha})\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\rho}_{\nu\alpha}(\delta\Gamma^{\alpha}_{\mu\lambda}). \tag{16}$$

Next we notice that

$$\nabla_\mu(\delta\Gamma^{\rho}_{\nu\lambda}) = \partial_\mu(\delta\Gamma^{\rho}_{\nu\lambda}) + \Gamma^{\rho}_{\mu\alpha}(\delta\Gamma^{\alpha}_{\nu\lambda}) - \Gamma^{\alpha}_{\nu\mu}(\delta\Gamma^{\rho}_{\alpha\lambda}) - \Gamma^{\alpha}_{\lambda\mu}(\delta\Gamma^{\rho}_{\alpha\nu}). \tag{17}$$

We can also calculate

$$\nabla_\nu(\delta\Gamma^\rho_{\mu\lambda}) = (\mu \leftrightarrow \nu) \tag{18}$$

Taking the difference between the above two objects gives

$$\nabla_\mu(\delta\Gamma^\rho_{\nu\lambda}) - \nabla_\nu(\delta\Gamma^\rho_{\mu\lambda}) = \partial_\mu(\delta\Gamma^\rho_{\nu\lambda}) - \partial_\nu(\delta\Gamma^\rho_{\mu\lambda}) + (\delta\Gamma^\rho_{\mu\alpha})\Gamma^\alpha_{\nu\lambda} + \Gamma^\rho_{\mu\alpha}(\delta\Gamma^\alpha_{\nu\lambda}) - (\delta\Gamma^\rho_{\nu\alpha})\Gamma^\alpha_{\mu\lambda} - \Gamma^\rho_{\nu\alpha}(\delta\Gamma^\alpha_{\mu\lambda}) \tag{19}$$

$$= \delta R^\rho{}_{\lambda\mu\nu}. \tag{20}$$

Next we can find the variation in the Ricci tensor by taking the trace of the variation of the Riemann tensor

$$\delta R_{\mu\nu} = \delta R^\lambda{}_{\mu\lambda\nu} = \nabla_\lambda(\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu(\delta\Gamma^\lambda_{\lambda\mu}). \tag{21}$$

Contracting this variation with the metric gives

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\lambda\big[g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu}\big], \tag{22}$$

and is a total derivative term which we can normally throw out. The issue becomes when we have a term e.g. a scalar field that couples to the Ricci scalar. Now we find δS takes the form

$$\delta S = \frac{1}{\kappa^2}\int \mathrm{d}^4x\left(\sqrt{-g}R_{\mu\nu}\delta g^{\mu\nu} - \sqrt{-g}\frac{1}{2}Rg_{\mu\nu}\delta g^{\mu\nu}\right) + \delta S_m \tag{23}$$

Recall that the functional derivative of the action satisfies

$$\delta S = \int \mathrm{d}^n x \sum_i \left(\frac{\delta S}{\delta\Phi^i}\delta\Phi^i\right), \tag{24}$$

where Φ^i is a complete set of fields being varied. This brings the total action δS to be

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \tag{25}$$

Defining the energy momentum tensor to be

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{26}$$

and lastly moving it to the other side we get

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{27}$$

2 $f(R)$ Gravity

Here we would like to derive the equations of motion for $f(R)$ gravity. In the interest of simplicity, we present the action as

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}f(R), \tag{28}$$

where of course $f(R)$ is some general function of the curvature scalar. Again we wish to vary the action with respect to to inverse metric $g^{\mu\nu}$. Proceeding accordingly shows

$$\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,[\sqrt{-g}\delta f(R) + f(R)\delta\sqrt{-g}] = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\left[\sqrt{-g}\frac{\delta f(R)}{\delta R}\delta R - \frac{1}{2}f(R)\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\right]. \tag{29}$$

f being some general scalar function of the curvature means the functional derivative should reduce down to just the normal derivative

$$\frac{\delta f}{\delta R} = \frac{\mathrm{d}f}{\mathrm{d}R} \equiv f'(R). \tag{30}$$

Next we need only to worry about varying the curvature scalar now

$$\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,[\sqrt{-g}f'(R)(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}) - \frac{1}{2}\sqrt{-g}f(R)g_{\mu\nu}\delta g^{\mu\nu}]. \tag{31}$$

We've already dealt with the variation in the Ricci tensor, but only to a certain extent. Since it was already a total derivative term, we could safely ignore it. Now however, we have a coupling between our arbitrary function and the curvature tensor which will yield non-trivial dynamics. Recall that the variation in the Ricci tensor is given by

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\lambda\big[g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\mu\nu}\big]. \tag{32}$$

Now we need to find out what the variation in the connection is. First we write

$$\Gamma_{\rho\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}), \tag{33}$$

where we can write

$$\Gamma^\lambda_{\mu\nu} = g^{\lambda\rho}\Gamma_{\rho\mu\nu}. \tag{34}$$

The variation in the connection can then be shown to be

$$\begin{aligned}
\delta\Gamma^\lambda_{\mu\nu} &= \delta g^{\lambda\rho}\Gamma_{\rho\mu\nu} + g^{\lambda\rho}\delta\Gamma_{\rho\mu\nu} && (35)\\
&= -g^{\alpha\lambda}g^{\beta\rho}\delta g_{\alpha\beta}\Gamma_{\rho\mu\nu} + \frac{1}{2}g^{\lambda\rho}(\partial_\mu\delta g_{\nu\rho} + \partial_\nu\delta g_{\mu\rho} - \partial_\rho\delta g_{\mu\nu}) && (36)\\
&= -g^{\lambda\rho}\Gamma^\beta_{\mu\nu}\delta g_{\beta\rho} + \frac{1}{2}g^{\lambda\rho}(\partial_\mu\delta g_{\nu\rho} + \partial_\nu\delta g_{\mu\rho} - \partial_\rho\delta g_{\mu\nu}) && (37)\\
&= \frac{1}{2}g^{\lambda\rho}(\partial_\mu\delta g_{\nu\rho} + \partial_\nu\delta g_{\mu\rho} - \partial_\rho\delta g_{\mu\nu} - 2\Gamma^\beta_{\mu\nu}\delta g_{\beta\rho}). && (38)
\end{aligned}$$

Next we introduce the terms $\pm\Gamma^\beta_{\mu\rho}\delta g_{\beta\nu}$ and $\pm\Gamma^\beta_{\nu\rho}\delta g_{\beta\mu}$ to the top to get

$$\delta\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu\delta g_{\nu\rho} - \Gamma^\beta_{\mu\nu}\delta g_{\beta\rho} - \Gamma^\beta_{\mu\rho}\delta g_{\beta\nu} + \partial_\nu\delta g_{\mu\rho} - \Gamma^\beta_{\nu\mu}\delta g_{\beta\rho} - \Gamma^\beta_{\nu\rho}\delta g_{\beta\mu} - \partial_\rho\delta g_{\mu\nu} + \Gamma^\beta_{\rho\mu}\delta g_{\beta\nu} + \Gamma^\beta_{\rho\nu}\delta g_{\beta\mu}) \tag{39}$$

$$= \frac{1}{2}g^{\lambda\rho}(\nabla_\mu\delta g_{\nu\rho} + \nabla_\nu\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\nu}). \tag{40}$$

This result implies we can write the variation in the Ricci tensor as

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{2}(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho})\nabla_\lambda(\nabla_\mu\delta g_{\nu\rho} + \nabla_\nu\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\nu}). \tag{41}$$

Plugging this into the action and integrating by parts gives us

$$\begin{aligned}
\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,[\sqrt{-g}(f'(R)R_{\mu\nu}\delta g^{\mu\nu} - \frac{1}{2}(g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho})[\nabla_\lambda f'(R)](\nabla_\mu\delta g_{\nu\rho} + \nabla_\nu\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\nu}))\\
- \frac{1}{2}\sqrt{-g}f(R)g_{\mu\nu}\delta g^{\mu\nu}].
\end{aligned} \tag{42}$$

We can integrate by parts again and manipulate the indices to get

$$\delta S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\Big[f'(R)R_{\mu\nu} - \nabla_\mu\nabla_\nu f'(R) + \Box f'(R)g_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}\Big]\delta g^{\mu\nu}. \tag{43}$$

Lastly, we can set the integral to zero and divide through by the variation and the volume element to get

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = f'(R)R_{\mu\nu} - (\nabla_\mu \nabla_\nu f'(R) - \Box f'(R) g_{\mu\nu}) - \frac{1}{2} f(R) g_{\mu\nu} = 0. \tag{44}$$

We can check if our answer is correct by setting $f(R) = R$ and we see that

$$f'(R) = 1 \Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0, \tag{45}$$

is the original Einstein equations in a vacuum. The equations of motion offer themselves a constraint on what the scalar curvature has to be by simply taking the trace

$$R = \frac{2f(R)}{f'(R)} - \frac{3\Box f'(R)}{f'(R)} \tag{46}$$

We can eliminate the covariant derivatives acting on the scalaron by making judicious use of the chain rule to get

$$-\frac{1}{2} f g_{\mu\nu} + f_R R_{\mu\nu} - f_{RR}\left[\nabla_\mu \nabla_\nu R - \nabla^2 R g_{\mu\nu}\right] - f_{RRR}\left[\nabla_\mu R \nabla_\nu R - (\nabla R)^2 g_{\mu\nu}\right] = 0, \tag{47}$$

where

$$f_R \equiv \frac{\mathrm{d}f}{\mathrm{d}R}, \quad f_{RR} \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}R^2}, \quad \ldots \tag{48}$$

and $(\nabla R)^2 \equiv g^{\mu\nu} \nabla_\mu R \nabla_\nu R$ and $\nabla^2 \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$.