Counting the Degrees of Freedom in Linearized General Relativity

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Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = 1$. The d'Alembert and Laplace operators are defined to be $\Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \nabla^2$ and $\nabla^2 = \partial_i \partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

1 Degrees of Freedom Overview

We are interested in the degrees of freedom for a given Lagrangian because in field theory, degrees of freedom correspond to a particle i.e. the force carrier for the field. A degree of freedom, broadly speaking, is an independent function (in our case) of spacetime coordinates. First we consider a scalar field $f : \mathbb{R}^4 \to \mathbb{R}$ usually denoted as $f(t, \mathbf{r})$. We say that f only carries one degree of freedom because the only independent function it carries is itself. Next we have a 4-vector field $V : \mathbb{R}^4 \to \mathbb{R}^4$ denoted by

$$V_\mu = (V_0, V_i). \tag{1}$$

Since V_0 is a scalar field, it carries a single degree of freedom like f. However the 3-vector field V_i is different because it is an array of three scalar fields and thus it carries three degrees of freedom. We can further decompose this vector field by using the following theorem from linear algebra:

Theorem 1.1 *Let X, Y, Z be vector spaces, and T:X → Y, U:Y → Z be linear. If UT:X → Z is invertible, then* $Y = \ker(U) \oplus \mathrm{Im}(T)$.

Taking Y to be the space of all vector fields $\mathcal{V}$, we can decompose it into two subspaces: the kernel of the divergence operator i.e. $\partial_i V^i = 0$ and the image of the gradient operator i.e. $\partial_i v$. Thus, any function[1] in $\mathcal{V}$ can be represented as

$$V_i = V_i^T + \partial_i v. \tag{2}$$

Since v is a scalar function, it propagates only a single degree of freedom. Naively, since V_i^T is a 3 component object, we would say it has 3 degrees of freedom. However, since we can "solve" for one of the components (and provided the fields go to zero at infinity) we can see that

$$\partial_x V_x^T + \partial_y V_y^T + \partial_z V_z^T = 0 \tag{3}$$

$$\Rightarrow V_z^T(x, y, z) = -\int_{-\infty}^{z} \partial_x V_x^T(x, y, z') + \partial_y V_y^T(x, y, z')\mathrm{d}z'. \tag{4}$$

Therefore given some initial data, V_z^T is completely determined by the components V_x^T and V_y^T which implies that V_i^T has only two independent functions i.e. two degrees of freedom.

Lastly, we move on to discussing tensors. A tensor T_{ij} is an object that maps elements of a vector space to a basis. Generically, a 3x3 tensor has 9 components. However for this discussion, the tensors we will most be interested in are symmetric i.e. $T_{ij} = T_{ji}$. The decomposition for a (spatial) tensor is slightly different than that of a vector. Symmetric tensors can be split into the image of the map taking functions to their traces, $T \to \frac{1}{3}T\delta_{ij}$, the space of transverse traceless tensors i.e. $\partial^i T_{ij} = 0, T^i_{\ i} = 0$, and the image of V under the map $V_i \to \partial_i V_j + \partial_j V_i - \frac{2}{3}(\partial^k V_k)\delta_{ij}$. This is another application of the above

[1]Note that this theorem is merely a generalization of Helmholtz' theorem which states that any vector field (sufficiently smooth) can be written as the sum of a divergence-less part and a curl-less part.

theorem where X is the space of vectors, Y is the space of traceless symmetric tensors, and Z is again the space of vectors, T maps $V_i \rightarrow \partial_i V_j + \partial_j V_i - \frac{2}{3}(\partial^k V_k)\delta_{ij}$ and U maps tensors to their divergence. Putting this all together, while also keeping plugging in the decomposition for vectors as well, any symmetric tensor can be written as

$$T_{ij} = T_{ij}^{TT} + \partial_i V_j^T + \partial_j V_i^T + 2\left(\partial_i\partial_j \upsilon - \frac{1}{3}\nabla^2 \upsilon \delta_{ij}\right) + \frac{1}{3}T\delta_{ij}, \tag{5}$$

where we have the following constraints/conditions

$$\partial^i T_{ij}^{TT} = 0, \quad T^{TTi}{}_i = 0, \quad \partial_i V_T^i = 0, \quad T = \delta^{ij} T_{ij}. \tag{6}$$

And now we ask how many degrees of freedom does T_{ij}^{TT} propagate? Since it's symmetric that means it has at most 6 independent components. Once we take into account its traceless-ness, that kills off an additional degree of freedom, so it can only have at most 5. Lastly, once we incorporate the fact that T_{ij}^{TT} is divergence-less, we find another three degrees of freedom are killed off and thus we can conclude T_{ij}^{TT} only propagates two degrees of freedom. For clarity's sake, the existence of a degree of freedom indicates a particle for that field, but the number of degrees of freedom for a particular field corresponds to the number of polarization modes.

2 Gauge Transformations

Here we give an exhaustive list of all the gauge transformations for the components of the metric perturbation with gauge parameter A_μ. Firstly, how does the metric perturbation transform under the action of a gauge? Well we can see that $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7}$$

Now that's all well and good, but how do the individual components themselves transform? First we should decompose the gauge parameter as outlined in the previous section. We know that since $h_{\mu\nu}$ is a symmetric (0,2) tensor, under spatial rotations the 00 component is a scalar, the 0i component forms a 3-vector, and the ij component forms a symmetric spatial tensor. This allows us to decompose the metric perturbation into it's constituent parts. Now we write $h_{\mu\nu}$ as

$$
\begin{aligned}
h_{00} &= h^{00} = -2\Phi, \\
h_{0i} &= -h^0_i = w_i, \\
h &= h^\mu_\mu = \eta^{\mu\nu} h_{\mu\nu} = 2\Phi + \bar{h}, \\
h_{ij} &= h^{TT}_{ij} + \partial_i v^T_j + \partial_j v^T_i + 2\left(\partial_i \partial_j \Psi - \frac{1}{3}\nabla^2 \Psi \delta_{ij}\right) + \frac{1}{3}\bar{h}\delta_{ij}, \\
w_i &= w^T_i + \partial_i \Omega,
\end{aligned}
\tag{8}
$$

subject to the conditions where

$$
\partial^i h^{TT}_{ij} = 0, \quad \delta^{ij} h^{TT}_{ij} = 0, \quad \partial^i v^T_i = \partial^i w^T_i = 0, \tag{9}
$$

and $\bar{h} = \mathrm{Tr}[h_{ij}] \equiv \delta^{ij} h_{ij}$. Now we write

$$
A_\mu = (A_0, A^T_i + \partial_i \alpha). \tag{10}
$$

Now we can plug into the transformation law from above and we get the following transformation rules for each component:

$$
\Phi \to \Phi + \dot{A}_0, \tag{11}
$$

$$
w^T_i \to w^T_i - \dot{A}^T_i, \tag{12}
$$

$$
v^T_i \to v^T_i - A^T_i, \tag{13}
$$

$$
\Omega \to \Omega - A_0 + \dot{\alpha}, \tag{14}
$$

$$
\bar{h} \to \bar{h} - \nabla^2 \alpha, \tag{15}
$$

$$
\Psi \to \Psi - \alpha. \tag{16}
$$

Since there are 4 scalar fields and 2 scalar parameters, we expect 4 - 2 gauge invariant scalar fields. Likewise, since there are 2 vector fields and 1 vector gauge parameter than we expect 2 - 1 gauge invariant vector fields. From the above transformation laws we can define the following gauge invariant fields

$$J \equiv -\Phi - \dot{\Omega} + \ddot{\Psi}, \tag{17}$$

$$L \equiv \frac{1}{3}(\bar{h} - 2\nabla^2\Psi), \tag{18}$$

$$M_i \equiv w_i^T - \dot{v}_i^T. \tag{19}$$

3 Linearized General Relativity

We start off with the Lagrangian for linearized General Relativity given by

$$\mathcal{L} = \partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h. \tag{20}$$

Next we perform a 3+1 decomposition of the metric perturbation which brings the Lagrangian to the form

$$\begin{aligned}\mathcal{L} = &- 2\partial_i w_j \dot{h}^{ij} - \partial_i w_j \partial^j w^i + \partial_i h_{jk}\partial^j h^{ik} - \frac{1}{2}\dot{\bar{h}}^2 + 2\partial_i\Phi\partial^i\bar{h} + \frac{1}{2}(\partial_i\bar{h})^2 \\ &+ \frac{1}{2}(\dot{h}_{ij})^2 + (\partial_i w_j)^2 - \frac{1}{2}(\partial_i h_{jk})^2 - 2w^i\partial_i\dot{\bar{h}} - 2\partial_i h^{ij}\partial_j\Phi - \partial_i h^{ij}\partial_j\bar{h}.\end{aligned} \tag{21}$$

Under the action, the Lagrangian takes on the form

$$\begin{aligned}S = \int \mathrm{d}^4x \Big[&2w^i\partial_j\dot{h}^{ij} + w_i(\partial^i\partial_k w^k - \nabla^2 w^i) + \partial^j h_{jk}\partial_i h^{ik} + \frac{1}{2}\bar{h}\Box\bar{h} - 2\Phi\nabla^2\bar{h} \\ &+\frac{1}{2}h_{ij}\Box h^{ij} - 2w^i\partial_i\dot{\bar{h}} + 2\Phi\partial_i\partial_j h^{ij} + \bar{h}\partial_i\partial_j h^{ij}\Big].\end{aligned} \tag{22}$$

We can streamline the calculation a bit by recognizing that we can treat the spin 0, 1, and 2 terms separately (i.e. we can assume there are no cross terms between differing spins). From this we can split the action into three different sectors:

$$S = S_T + S_V + S_S, \tag{23}$$

where

$$S_T = \int -\frac{1}{2}h^{ij}_{TT}\ddot{h}^{TT}_{ij} + \frac{1}{2}h^{ij}_{TT}\nabla^2 h^{TT}_{ij}\,\mathrm{d}^4x\,, \tag{24}$$

$$
\begin{aligned}
S_V = &\int 2w_i^T \nabla^2 \dot{v}_T^i - w_i^T \nabla^2 w_T^i + \nabla^2 v_T^i \left(\nabla^2 v_i^T - \left(-\frac{\partial^2}{\partial t^2} + \nabla^2 \right) v_i^T \right) \mathrm{d}^4 x \,, \\
&= \int (\partial_i w_j^T - \partial_i \dot{v}_j^T)^2 \, \mathrm{d}^4 x \,,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
S_S = \int & 2\Omega \nabla^2 \dot{\bar{h}} - \frac{8}{3} \Omega \nabla^4 \dot{\Psi} - \frac{2}{3} \Omega \nabla^2 \dot{\bar{h}} - \frac{16}{9} \nabla^4 \Psi \nabla^2 \Psi - \frac{8}{9} \nabla^2 \Psi \nabla^2 \bar{h} \\
& - \frac{1}{9} \bar{h} \nabla^2 \bar{h} - \frac{1}{2} \bar{h} \Box \bar{h} + \frac{4}{3} \nabla^2 \Psi \Box \nabla^2 \Psi + \frac{1}{6} \bar{h} \Box \bar{h} - 2\Phi \nabla^2 \bar{h} \\
& + \frac{2}{3} \Phi \nabla^2 \bar{h} + \frac{8}{3} \Phi \nabla^4 \Psi + \frac{4}{3} \bar{h} \nabla^4 \Psi + \frac{1}{3} \bar{h} \nabla^2 \bar{h} \mathrm{d}^4 x.
\end{aligned} \tag{26}
$$

Inserting the gauge-invariant fields $J \equiv -\Phi - \dot{\Omega} + \ddot{\Psi}$, $L \equiv \frac{2}{3}(\bar{h} - 2\nabla^2 \Psi)$, and $M_i = w_i^T - \dot{v}_i^T$, S_S and S_V take on the forms

$$
S_V = \int \frac{1}{2} (\partial_i M_j)^2 \, \mathrm{d}^4 x \,, \tag{27}
$$

$$
S_S = \int 2J \nabla^2 L - \frac{1}{4} L \nabla^2 L + \frac{1}{2} L \ddot{L} \, \mathrm{d}^4 x \,, \tag{28}
$$

We can now analyze the true degrees of freedom that are present in $h_{\mu\nu}$. First, looking at the vector action we can see that no time derivatives of M_i are present in the action. Therefore it is an auxiliary field and we may use its equations of motion (EOM) to eliminate it. Proceeding accordingly we find

$$
\frac{\delta \mathcal{L}}{\delta M^i} = \nabla^2 M_i = 0 \Rightarrow M_i = 0, \tag{29}
$$

which implies that $S_V = 0$. Next we turn our attention to the scalar action. Since J appears linearly with no time derivatives, we may interpret it as a Lagrange multiplier. From there we can see that the EOM of J enforces the following constraint:

$$
\frac{\delta \mathcal{L}}{\delta J} = \nabla^2 L = 0 \Rightarrow L = 0, \tag{30}
$$

and therefore, $S_S = 0$. The total action is now

$$
S = S_T, \tag{31}
$$

where

$$S_T = \int \frac{1}{2} h^{ij}_{TT} \Box h^{TT}_{ij} \, \mathrm{d}^4 x \,. \tag{32}$$

Since we've finally eliminated all of the purely gauge fields we're left with just the tensor action. Since h^{TT}_{ij} carries 2 independent modes, we can finally conclude our analysis that linearized General Relativity carries with it a maximum of two degrees of freedom.