Hamiltonian Formulation of General Relativity

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We are interested in quantizing GR via the canonical quantization approach. To get to that point, there is quite a lot of ground we have to cover. First we write down the Einstein-Hilbert action

$$S = \frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}(\mathcal{R} - 2\Lambda) + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_{matter}, \tag{1}$$

where $\kappa^2 = M_P^{-2} = 8\pi G$, $\mathcal{R}$ is the scalar curvature on the full manifold, h is the determinant of the spatial metric , K is the trace of the extrinsic curvature and S_{matter} is the action for matter.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-, +, +, +)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_P = \frac{1}{\sqrt{8\pi G}} \approx 2.43 \times 10^{18}$ GeV. The d'Alembert and Laplace operators are defined to be $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$ and $\nabla^2 = h^{ij}\partial_i\partial_j$ respectively. We use boldface letters $\mathbf{x}$ to indicate 3-vectors and we use x and p to denote 4-vectors. Conventions for the curvature tensors, $\mathcal{R}^\lambda{}_{\rho\mu\nu}$, $\mathcal{R}_{\mu\nu}$, $\mathcal{R}$ covariant, ∇_μ, and Lie derivatives $\mathcal{L}_m$ are all taken from Carroll.

1 Geometry

We'll skip some of the gory differential geometric details and motivations for now and merely define a unit vector n^μ which satisfies $n^\mu n_\mu = -1$ and is normal to the hypersurface

Σ_t. Because we want to conceptualize this surface as representing *spatial* slices of constant times t, we require that Σ_t be a spacelike surface so that

$$g_{ij}V^iV^j > 0, \tag{2}$$

so that our norms are positive definite (i.e. length is always positive). This justifies the constraint we place on our unit vector as being timelike (Note: our metric signature is $(-,+,+,+)$). Since all of our calculations will be done on the hypersurface Σ_t, we want to make sure that we're only dealing with vectors that are not parallel to n^μ. Thus we define the projection tensor

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{3}$$

Note for arbitrary v that lives on the tangent space of Σ we have

$$h_{\mu\nu}V^\nu = g_{\mu\nu}V^\nu = V_\mu, \tag{4}$$

and thus we can also refer to the projection tensor as the induced, or spatial metric on the tangent space of Σ i.e. $\mathcal{T}_p(\Sigma)$. If we define the unit normal vector as

$$n_\mu = -N\nabla_\mu t, \tag{5}$$

where N is the lapse function, then the components of the spatial metric reduces down to $h_{ij} = g_{ij}$. Note

$$h_{\mu\nu}n^\nu = (g_{\mu\nu} + n_\mu n_\nu)n^\nu = n_\mu - n_\mu = 0, \tag{6}$$

and that

$$g^{\mu\lambda}h_{\lambda\nu} = g^{\mu\lambda}(g_{\lambda\nu} + n_\lambda n_\nu) = \delta^\mu{}_\nu + n^\mu n_\nu. \tag{7}$$

Now we define the extrinsic curvature $K_{\mu\nu}$ as being

$$-K_{\mu\nu} = \nabla_\nu n_\mu + a_\mu n_\nu, \tag{8}$$

where $a^\mu = n^\lambda \nabla_\lambda n^\mu$. We can see that

$$a^\mu n_\mu = n_\mu n^\lambda \nabla_\lambda n^\mu = \frac{1}{2} n^\lambda \nabla_\lambda (n^\mu n_\mu) = 0, \tag{9}$$

and also $K = -\nabla_\mu n^\mu$. Since we have both a metric and an object to characterize how Σ bends on $\mathcal{M}$, we can define a covariant derivative $D_\mu \equiv h^\lambda{}_\mu \nabla_\lambda$. From the covariant derivative on Σ, we can define the Riemannian (i.e. the intrinsic curvature) by assuming its definition is similar to the case of the base manifold i.e.

$$[D_\mu, D_\nu] V^\lambda = R^\lambda{}_{\rho\mu\nu} V^\rho, \tag{10}$$

where $[\cdot, \cdot]$ is the usual commutator as defined in quantum mechanics. Next we will relate the intrinsic curvature on Σ to the intrinsic curvature on $\mathcal{M}$. Let us focus on the $D_\mu D_\nu V^\lambda$ term first

$$\begin{aligned}
D_\mu(D_\nu V^\lambda) &= h^\lambda{}_\sigma h^\alpha{}_\mu h^\beta{}_\nu \nabla_\alpha (D_\beta V^\sigma) && (11)\\
&= h^\lambda{}_\sigma h^\alpha{}_\mu h^\beta{}_\nu \nabla_\alpha (h^\gamma{}_\beta \nabla_\gamma h^\sigma{}_\rho V^\rho) && (12)\\
&= h^\lambda{}_\sigma h^\alpha{}_\mu h^\beta{}_\nu h^\sigma{}_\rho h^\gamma{}_\beta \nabla_\alpha \nabla_\gamma V^\rho + h^\lambda{}_\sigma h^\alpha{}_\mu h^\beta{}_\nu h^\sigma{}_\rho (\nabla_\alpha h^\gamma{}_\beta) \nabla_\gamma V^\rho + h^\lambda{}_\sigma h^\alpha{}_\mu h^\beta{}_\nu h^\gamma{}_\beta (\nabla_\alpha h^\sigma{}_\rho) \nabla_\gamma V^\rho && (13)\\
&= h^\alpha{}_\mu h^\beta{}_\nu h^\lambda{}_\rho h^\gamma{}_\beta \nabla_\alpha \nabla_\gamma V^\rho + h^\alpha{}_\mu h^\beta{}_\nu h^\lambda{}_\rho (\nabla_\alpha h^\gamma{}_\beta) \nabla_\gamma V^\rho + h^\lambda{}_\sigma h^\alpha{}_\mu h^\beta{}_\nu h^\gamma{}_\beta (\nabla_\alpha h^\sigma{}_\rho) \nabla_\gamma V^\rho. && (14)
\end{aligned}$$

Next we'll focus on the last two terms in the last line. The covariant derivative when acting on the spatial metric is

$$\begin{aligned}
h^\alpha{}_\mu h^\beta{}_\nu \nabla_\gamma V^\rho (h^\lambda{}_\rho \nabla_\alpha h^\gamma{}_\beta + h^\lambda{}_\sigma h^\gamma{}_\beta \nabla_\alpha h^\sigma{}_\rho) &= h^\alpha{}_\mu h^\beta{}_\nu \nabla_\gamma V^\rho (h^\lambda{}_\rho \nabla_\alpha (n^\gamma n_\beta) + h^\lambda{}_\sigma h^\gamma{}_\beta \nabla_\alpha (n^\sigma n_\rho)) && (15)\\
&= h^\alpha{}_\mu h^\beta{}_\nu \nabla_\gamma V^\rho [h^\lambda{}_\rho (n_\beta \nabla_\alpha n^\gamma + n^\gamma \nabla_\alpha n_\beta) + h^\lambda{}_\sigma h^\gamma{}_\beta n_\rho \nabla_\alpha n^\sigma]. && (16)
\end{aligned}$$

Remembering that $\nabla_\nu n_\mu = -K_{\mu\nu} - a_\mu n_\nu$, the above line becomes

$$h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}\nabla_{\gamma}V^{\rho}[h^{\lambda}{}_{\rho}(n_{\beta}\nabla_{\alpha}n^{\gamma}+n^{\gamma}\nabla_{\alpha}n_{\beta})+h^{\lambda}{}_{\sigma}h^{\gamma}{}_{\beta}n_{\rho}\nabla_{\alpha}n^{\sigma}]=-n^{\gamma}h^{\lambda}{}_{\rho}h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}K_{\beta\alpha}\nabla_{\gamma}V^{\rho}-n_{\rho}h^{\gamma}{}_{\nu}K^{\lambda}{}_{\mu}\nabla_{\lambda}V^{\rho} \tag{17}$$

$$=-n^{\gamma}h^{\lambda}{}_{\rho}K_{\mu\nu}\nabla_{\gamma}V^{\rho}-n_{\rho}h^{\gamma}{}_{\nu}K^{\lambda}{}_{\mu}\nabla_{\lambda}V^{\rho}. \tag{18}$$

We can do a partial integration on the second term $\nabla_{\gamma}(V^{\rho}n_{\rho})=n_{\rho}\nabla_{\gamma}V^{\rho}+V^{\rho}\nabla_{\gamma}n_{\rho}$ and remembering $V^{\rho}n_{\rho}=0$ which brings us

$$n^{\gamma}h^{\lambda}{}_{\rho}K_{\mu\nu}\nabla_{\gamma}V^{\rho}-n_{\rho}h^{\gamma}{}_{\nu}K^{\lambda}{}_{\mu}\nabla_{\gamma}V^{\rho}=-h^{\lambda}{}_{\rho}K_{\mu\nu}n^{\gamma}\nabla_{\gamma}V^{\rho}+h^{\gamma}{}_{\nu}K^{\lambda}{}_{\mu}K_{\gamma\rho}V^{\rho} \tag{19}$$

$$=K^{\lambda}{}_{\mu}K_{\rho\nu}V^{\rho}-K_{\mu\nu}n^{\gamma}\nabla_{\gamma}V^{\lambda}. \tag{20}$$

The $D_{\nu}D_{\mu}V^{\lambda}$ expression is gotten by simply $(\mu\leftrightarrow\nu)$ which implies that the Riemann tensor is just

$$R^{\lambda}{}_{\rho\mu\nu}V^{\rho}=h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}h^{\lambda}{}_{\rho}(\nabla_{\alpha}\nabla_{\beta}-\nabla_{\beta}\nabla_{\alpha})V^{\rho}-(K^{\lambda}{}_{\mu}K_{\rho\nu}-K^{\lambda}{}_{\nu}K_{\mu\rho})V^{\rho} \tag{21}$$

$$=h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}h^{\lambda}{}_{\rho}\mathcal{R}^{\rho}{}_{\pi\alpha\beta}V^{\pi}-(K^{\lambda}{}_{\mu}K_{\rho\nu}-K^{\lambda}{}_{\nu}K_{\mu\rho})V^{\rho}. \tag{22}$$

Thus, the Riemann curvature on the foliated hypersurface is simply

$$R^{\lambda}{}_{\rho\mu\nu}=h^{\lambda}{}_{\sigma}h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}h^{\pi}{}_{\rho}\mathcal{R}^{\sigma}{}_{\pi\alpha\beta}-(K^{\lambda}{}_{\mu}K_{\rho\nu}-K^{\lambda}{}_{\nu}K_{\mu\rho}) \tag{23}$$

Next we can contract the first and third indices to get the Ricci curvature tensor

$$R_{\mu\nu}=R^{\lambda}{}_{\mu\lambda\nu}=h^{\alpha}{}_{\sigma}h^{\beta}{}_{\nu}h^{\pi}{}_{\mu}\mathcal{R}^{\sigma}{}_{\pi\alpha\beta}-(KK_{\mu\nu}-K^{\lambda}{}_{\nu}K_{\mu\lambda}) \tag{24}$$

$$=h^{\alpha}{}_{\mu}h^{\beta}{}_{\nu}\mathcal{R}_{\alpha\beta}+n^{\alpha}n_{\sigma}h^{\pi}{}_{\mu}h^{\beta}{}_{\nu}\mathcal{R}^{\sigma}{}_{\pi\alpha\beta}-(KK_{\mu\nu}-K_{\mu\lambda}K^{\lambda}{}_{\nu}). \tag{25}$$

Contracting with the spatial tensor this time, the curvature scalar is simply

$$R=h^{\mu\nu}R_{\mu\nu}=(g^{\alpha}{}_{\sigma}+n^{\alpha}n_{\sigma})h^{\beta\pi}\mathcal{R}^{\sigma}{}_{\pi\alpha\beta}-(K^2-K^{\mu\nu}K_{\mu\nu}) \tag{26}$$

$$=(g^{\beta\pi}+n^{\beta}n^{\pi})R_{\pi\beta}-h^{\beta\pi}n^{\alpha}n_{\sigma}\mathcal{R}^{\sigma}{}_{\pi\alpha\beta}-(K^2-K^{\mu\nu}K_{\mu\nu}) \tag{27}$$

$$=\mathcal{R}+2\mathcal{R}_{\mu\nu}n^{\mu}n^{\nu}-(K^2-K^{\mu\nu}K_{\mu\nu}). \tag{28}$$

Now that we've defined all of our geometric objects, we can study how these objects, particularly the objects that are related to the extrinsic curvature of the hypersurface, evolve as we transport a vector across the surface. The acceleration $a^\mu = n^\nu \nabla_\nu n^\mu$ can be written as

$$a^\mu = n^\nu \nabla_\nu n^\mu = -n^\nu \nabla_\nu (N \nabla^\mu t) = -n^\nu \nabla_\nu N \nabla^\mu t - N n^\nu \nabla_\nu \nabla^\mu t \tag{29}$$

$$= \frac{n^\nu}{N} n^\mu \nabla_\nu N + n^\nu N \nabla^\mu \left(\frac{n_\nu}{N}\right) = \frac{n^\mu n^\nu}{N} \nabla_\nu N + n^\nu \nabla^\mu n_\nu - N \nabla^\mu \left(\frac{1}{N}\right) \tag{30}$$

$$= \frac{n^\mu n^\nu}{N} \nabla_\nu N + \frac{1}{N} \nabla^\mu N = \frac{1}{N} (\nabla^\mu N + n^\mu n^\nu \nabla_\nu N) \tag{31}$$

$$= \frac{1}{N} (g^{\mu\nu} + n^\mu n^\nu) \nabla_\nu N = \frac{1}{N} h^{\mu\nu} \nabla_\nu N = D^\mu \ln N. \tag{32}$$

Next we can check how the normal evolution vector changes as its parallel transported across Σ_t

$$\nabla_\nu m_\mu = \nabla_\nu (N n_\mu) = n_\mu \nabla_\nu N + N \nabla_\nu n_\mu = n_\mu \nabla_\nu N - N K_{\mu\nu} - N a_\mu n_\nu \tag{33}$$

$$= n_\mu \nabla_\nu N - N K_{\mu\nu} - n_\nu D_\mu N. \tag{34}$$

We now have all the tools to evaluate the evolution of the spatial metric:

$$\mathcal{L}_m h_{\mu\nu} = m^\lambda \nabla_\lambda h_{\mu\nu} + h_{\mu\lambda} \nabla_\nu m^\lambda + h_{\lambda\nu} \nabla_\mu m^\lambda \tag{35}$$

$$= m^\lambda \nabla_\lambda h_{\mu\nu} + h_{\mu\lambda} (n^\lambda \nabla_\nu N - N K^\lambda{}_\nu - n_\nu D^\lambda N) + h_{\lambda\nu} (n^\lambda \nabla_\mu N - N K^\lambda{}_\mu - n_\mu D^\lambda N) \tag{36}$$

$$= N n^\lambda \nabla_\lambda (n_\mu n_\nu) - 2N K_{\mu\nu} - n_\nu D_\mu N - n_\mu D_\nu N \tag{37}$$

$$= N a_\mu n_\nu + N a_\nu n_\mu - 2N K_{\mu\nu} - n_\nu D_\mu N - n_\mu D_\nu N \tag{38}$$

$$= N n_\nu D_\mu \ln N + N n_\mu D_\nu \ln N - 2N K_{\mu\nu} - n_\nu D_\mu N - n_\mu D_\nu N \tag{39}$$

$$= -2N K_{\mu\nu}. \tag{40}$$

We can also see

$$\mathcal{L}_m h_{\mu\nu} = \mathcal{L}_{Nn} h_{\mu\nu} = Nn^\lambda \nabla_\lambda h_{\mu\nu} + Nh_{\mu\lambda}\nabla_\nu n^\lambda + h_{\mu\lambda} n^\lambda \nabla_\nu N + Nh_{\nu\lambda}\nabla_\mu n^\lambda + h_{\nu\lambda} n^\lambda \nabla_\mu N \tag{41}$$

$$= Nn^\lambda \nabla_\lambda h_{\mu\nu} + Nh_{\mu\lambda}\nabla_\nu n^\lambda + Nh_{\nu\lambda}\nabla_\mu n^\lambda \tag{42}$$

$$= N\mathcal{L}_n h_{\mu\nu}. \tag{43}$$

Next we compute the evolution of $h^\mu{}_\nu$

$$\mathcal{L}_m h^\mu{}_\nu = m^\lambda \nabla_\lambda h^\mu{}_\nu - h^\lambda{}_\nu \nabla_\lambda m^\mu + h^\mu{}_\lambda \nabla_\nu m^\lambda \tag{44}$$

$$= m^\lambda \nabla_\lambda (n^\mu n_\nu) - h^\lambda{}_\nu \nabla_\lambda (Nn^\mu) + h^\mu{}_\lambda \nabla_\nu (Nn^\lambda) \tag{45}$$

$$= Nn_\nu n^\lambda \nabla_\lambda n^\mu + Nn^\mu n^\lambda \nabla_\lambda n_\nu - Nh^\lambda{}_\nu \nabla_\lambda n^\mu - n^\mu h^\lambda{}_\nu \nabla_\lambda N + Nh^\mu{}_\lambda \nabla_\nu n^\lambda \tag{46}$$

$$= Nn_\nu a^\mu + Nn^\mu a_\nu - N(g^\lambda{}_\nu + n^\lambda n_\nu)\nabla_\lambda n^\mu - n^\mu (g^\lambda{}_\nu + n^\lambda n_\nu)\nabla_\lambda N + N(g^\mu{}_\lambda + n^\mu n_\lambda)\nabla_\nu n^\lambda \tag{47}$$

$$= n_\nu D^\mu N - Nn_\nu n^\lambda \nabla_\lambda n^\mu = n_\nu D^\mu N - Nn_\nu a^\mu = 0, \tag{48}$$

so we can use $h^\mu{}_\nu$ to raise and lower indices for tensors that are being acted on by the Lie derivative. Let us calculate the Lie derivative of the extrinsic curvature

$$\mathcal{L}_m K_{\mu\nu} = m^\lambda \nabla_\lambda K_{\mu\nu} + K_{\mu\lambda}\nabla_\nu m^\lambda + K_{\lambda\nu}\nabla_\mu m^\lambda \tag{49}$$

$$= Nn^\lambda \nabla_\lambda K_{\mu\nu} + K_{\mu\lambda}(n^\lambda \nabla_\nu N - NK^\lambda{}_\nu - n_\nu D^\lambda N) + K_{\lambda\nu}(n^\lambda \nabla_\mu N - NK^\lambda{}_\mu - n_\mu D^\lambda N) \tag{50}$$

$$= Nn^\lambda \nabla_\lambda K_{\mu\nu} + n^\lambda (K_{\mu\lambda}\nabla_\nu N + K_{\lambda\nu}\nabla_\mu N) - 2NK_{\mu\lambda}K^\lambda{}_\nu - (n_\nu K_{\mu\lambda} + n_\mu K_{\lambda\nu})D^\lambda N \tag{51}$$

$$= Nn^\lambda \nabla_\lambda K_{\mu\nu} - 2NK_{\mu\lambda}K^\lambda{}_\nu - (n_\nu K_{\mu\lambda} + n_\mu K_{\lambda\nu})D^\lambda N. \tag{52}$$

It'll be useful for us when we want to write down the Hamiltonian for GR to compute $h_{\mu\lambda} n^\sigma h^\rho{}_\nu \mathcal{R}^\lambda{}_{\rho\mu\nu} n^\rho$

$$
\begin{aligned}
h_{\mu\lambda} n^\sigma h^\rho{}_\nu \mathcal{R}^\lambda{}_{\rho\mu\nu} n^\rho &= h_{\mu\lambda} n^\sigma h^\rho{}_\nu (\nabla_\rho \nabla_\sigma - \nabla_\sigma \nabla_\rho) n^\lambda && (53) \\
&= h_{\mu\lambda} n^\sigma h^\rho{}_\nu [\nabla_\sigma (K^\lambda{}_\rho + n_\rho D^\lambda \ln N) - \nabla_\rho (K^\lambda{}_\sigma + n_\sigma D^\lambda \ln N)] && (54) \\
&= h_{\mu\lambda} h^\rho{}_\nu n^\sigma [\nabla_\sigma K^\lambda{}_\rho - \nabla_\rho K^\lambda{}_\sigma + (a_\sigma n_\rho - a_\rho n_\sigma) D^\lambda \ln N - n_\sigma \nabla_\rho D^\lambda \ln N] && (55) \\
&= h_{\mu\lambda} h^\rho{}_\nu [n^\sigma \nabla_\sigma K^\lambda{}_\rho + K^\lambda{}_\sigma \nabla_\rho n^\sigma + n_\rho n^\sigma \nabla_\sigma D^\lambda \ln N + \nabla_\rho D^\lambda \ln N + (D_\rho \ln N)(D^\lambda \ln N && (56) \\
&= h_{\mu\lambda} h^\rho{}_\nu n^\sigma \nabla_\sigma K^\lambda{}_\rho - K_{\mu\lambda} K^\lambda{}_\nu + h_{\mu\lambda} D_\nu D^\lambda \ln N + h_{\mu\lambda} h^\rho{}_\nu (D_\rho \ln N)(D^\lambda \ln N) && (57) \\
&= h_{\mu\lambda} h^\rho{}_\nu n^\sigma \nabla_\sigma K^\lambda{}_\rho - K_{\mu\lambda} K^\lambda{}_\nu + \frac{1}{N} D_\nu D_\mu N. && (58)
\end{aligned}
$$

Recall the Lie derivative along the direction of the normal evolution vector of the extrinsic curvature is

$$
\mathcal{L}_m K_{\mu\nu} = 2N K_{\mu\lambda} K^\lambda{}_\nu - (n_\nu K_{\mu\lambda} + n_\mu K_{\lambda\nu}) D^\lambda N. \tag{59}
$$

We can express the contracted Riemann tensor as

$$
h_{\mu\lambda} n^\sigma h^\rho{}_\nu \mathcal{R}^\lambda{}_{\rho\mu\nu} n^\rho = \frac{1}{N} \mathcal{L}_m K_{\mu\nu} + K_{\mu\lambda} K^\lambda{}_\nu + \frac{1}{N} D_\nu D_\mu N. \tag{60}
$$

This brings equation 24 to the form

$$
h^\alpha{}_\mu h^\beta{}_\nu \mathcal{R}_{\alpha\beta} = R_{\mu\nu} - \frac{1}{N} \mathcal{L}_m K_{\mu\nu} - 2K_{\mu\lambda} K^\lambda{}_\nu - \frac{1}{N} D_\nu D_\mu N + K K_{\mu\nu}. \tag{61}
$$

Taken the trace with the spatial metric is

$$
\begin{aligned}
h^{\mu\nu} \mathcal{R}_{\mu\nu} = \mathcal{R} + \mathcal{R}_{\mu\nu} n^\mu n^\nu &= R + K^2 - K^{\mu\nu} K_{\mu\nu} - \frac{1}{N} h^{\mu\nu} \mathcal{L}_m K_{\mu\nu} - \frac{1}{N} D_\mu D^\mu N && (62) \\
&= R + K^2 - K^{ij} K_{ij} - \frac{1}{N} h^{ij} \mathcal{L}_m K_{ij} - \frac{1}{N} D_i D^i N, && (63)
\end{aligned}
$$

where we swap the spacetime indices for spatial indices because all of the objects on the righthand side are spatial. Now we'll focus on the Lie derivative term

$$
h^{ij} \mathcal{L}_m K_{ij} = \mathcal{L}_m (h^{ij} K_{ij}) - K_{ij} \mathcal{L}_m h^{ij} = \mathcal{L}_m K - K_{ij} \mathcal{L}_m h^{ij}. \tag{64}
$$

Now let's calculate the Lie derivative of the inverse spatial metric

$$h^{ik}h_{kj} = \delta^i{}_j \Rightarrow h^{ik}\mathcal{L}_m h_{kj} + h_{kj}\mathcal{L}_m h^{ik} = 0 \Rightarrow \mathcal{L}_m h^{i\ell} = -h^{ik}h^{j\ell}\mathcal{L}_m h_{kj} = 2Nh^{ik}h^{j\ell}K_{kj}. \tag{65}$$

Thus $\mathcal{L}_m h^{ij} = 2NK^{ij}$. Recalling $\mathcal{R}_{\mu\nu}n^\mu n^\nu = \frac{1}{2}(R + K^2 - K^{ij}K_{ij} - \mathcal{R})$, plugging this and the previous result into equation (63) gives us

$$\mathcal{R} = R + K^2 + K^{ij}K_{ij} - \frac{2}{N}\mathcal{L}_m K - \frac{2}{N}D_i D^i N. \tag{66}$$

2 The Hamiltonian Density

Let us compute the 4-metric. First we define

$$N \equiv (-g^{00})^{-\frac{1}{2}}, \quad g_{0i} \equiv N_i, \tag{67}$$

and recall $g_{ij} = h_{ij}$ where N_i is the shift vector. We can find the other components by using the following

$$\delta^\mu{}_\nu = g^{\mu\lambda}g_{\lambda\nu} = g^{\mu 0}g_{0\nu} + g^{\mu i}g_{i\nu}. \tag{68}$$

Now we have

$$\delta^0{}_0 = 1 = g^{00}g_{00} + g^{0i}g_{0i}, \quad \delta^0{}_i = 0 = g^{00}g_{0i} + g^{0k}g_{ki}. \tag{69}$$

From the second equation we can compute $g^0{}_i$ term

$$g^0{}_i = -g^{00}N_i \Rightarrow 1 = g^{00}g_{00} - g^{00}N^i N_i \Leftrightarrow g_{00} = -N^2 + N_i N^i. \tag{70}$$

A similar procedure to find the inverse metric yields

$$g^{ij} = h^{ij} - \frac{N^i N^j}{N^2}. \tag{71}$$

Lastly, the determinant of the spatial metric can be found by

$$g^{00} = \frac{C_{00}}{\det(g_{\mu\nu})} = \frac{C_{00}}{g}, \tag{72}$$

where $C_{\mu\nu} = (-1)^{\mu+\nu} M_{\mu\nu}$ is the co factor matrix and $M_{\mu\nu}$ is the determinant of the metric with the 0-th column and row deleted which leaves

$$g^{00} = \frac{h}{g} \Rightarrow g = \frac{h}{g^{00}} = -N^2 h. \tag{73}$$

And so $\sqrt{-g} = N\sqrt{h}$. Finally, we can decompose the Lie derivative operator along the direction of the normal evolution as $\mathcal{L}_m = \mathcal{L}_{\partial_t} - \mathcal{L}_N$. This brings the extrinsic curvature to the form

$$K_{ij} = -\frac{1}{2N}\mathcal{L}_m h_{ij} = -\frac{1}{2N}(\mathcal{L}_{\partial_t} h_{ij} - \mathcal{L}_N h_{ij}) = \frac{1}{2N}\Big(-\dot{h}_{ij} + D_i N_j + D_j N_i\Big). \tag{74}$$

We finally have all the ingredients to write the Einstein-Hilbert action on 3-space:

$$S = \frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\mathcal{R} - 2\Lambda + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_{matter} \tag{75}$$

$$= \frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h}[R - 2\Lambda + K^2 + K^{ij}K_{ij} - \frac{2}{N}\mathcal{L}_m K - \frac{2}{N}D_i D^i N] + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_{matter}. \tag{76}$$

The Lie derivative of the trace of the extrinsic curvature is

$$\mathcal{L}_m K = m^\lambda \nabla_\lambda K = N n^\mu \nabla_\mu K = N\nabla_\mu(Kn^\mu) - NK\nabla_\mu n^\mu = N\nabla_\mu(Kn^\mu) + NK^2. \tag{77}$$

Which when plugged back into the action yields

$$S = \frac{1}{2\kappa^2}\left[\int_{\mathcal{M}} \mathrm{d}^4x N\sqrt{h}\Big[R - 2\Lambda + K^{ij}K_{ij} - K^2 - 2\nabla_\mu(Kn^\mu) - \frac{2}{N}D_i D^i N\Big] + 2\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}K\right] + S_{matter} \tag{78}$$

The divergence term can be taken care of via the following

$$-2\int_{\mathcal{M}} N\sqrt{h}\nabla_\mu(Kn^\mu)\mathrm{d}^4x = -2\int_{\mathcal{M}} \sqrt{-g}\nabla_\mu(Kn^\mu)\mathrm{d}^4x \tag{79}$$

$$= -2\int_{\mathcal{M}} \partial_\mu(\sqrt{-g}Kn^\mu)\mathrm{d}^4x \tag{80}$$

$$= 2\int_{\partial\mathcal{M}} Kn^\mu n_\mu\sqrt{h}\mathrm{d}^3x \tag{81}$$

$$= -2\int_{\partial\mathcal{M}} K\sqrt{h}\mathrm{d}^3x, \tag{82}$$

which exactly cancels out with the Gibbons-Hawking-York term. Since D_iD^iN is a total divergence term, we can discard it from the action. This simplifies the action immensely to the form

$$S = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^4xN\sqrt{h}[R - 2\Lambda + K^{ij}K_{ij} - K^2] + S_{matter}. \tag{83}$$

Now we can write down the Hamiltonian for GR. We'll need to find the conjugate momenta of the dynamical fields. In this case, the only dynamical field is the spatial metric h_{ij}. The Lagrangian can be written as

$$\mathcal{L} = N\sqrt{h}[R - \Lambda + (h^{ik}h^{j\ell} - h^{ij}h^{k\ell})K_{ij}K_{k\ell}]. \tag{84}$$

First we compute the partial derivative of the extrinsic curvature with respect to the spatial metric

$$\frac{\partial K_{ij}}{\partial \dot{h}_{k\ell}} = -\frac{1}{2N}\delta^k{}_i\delta^\ell{}_j. \tag{85}$$

Next we define the conjugate momenta of the spatial metric

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial\dot{h}_{ij}} \tag{86}$$

$$= N\sqrt{h}\left[(h^{mk}h^{n\ell} - h^{mn}h^{k\ell})\left(-\frac{1}{2N}\delta^i{}_m\delta^j{}_n\right)K_{k\ell} - \frac{1}{2N}(h^{mk}h^{n\ell} - h^{mn}h^{k\ell})\delta^i{}_k\delta^j{}_\ell K_{mn}\right] \tag{87}$$

$$= -\sqrt{h}(h^{ik}h^{j\ell} - h^{ij}h^{k\ell})K_{k\ell} \tag{88}$$

$$= \sqrt{h}(h^{ij}K - K^{ij}). \tag{89}$$

Now we shall invert this equation to solve for the extrinsic curvature tensor

$$\pi^{ij} = \sqrt{h}(h^{ij}K - K^{ij}) \Rightarrow h_{ij}\pi^{ij} = \sqrt{h}(\delta^i{}_i K - h_{ij}K^{ij}) = \sqrt{h}(3K - K) = 2\sqrt{h}K. \quad (90)$$

So $K = \frac{1}{2\sqrt{h}} h_{ij}\pi^{ij}$. Plugging this back into the definition of the conjugate momentum gives us

$$K^{ij} = \frac{1}{\sqrt{h}}\left(\frac{1}{2}h^{ij}h_{k\ell}\pi^{k\ell} - \pi^{ij}\right) \Rightarrow K_{ij} = \frac{1}{\sqrt{h}}\left(\frac{1}{2}h_{ij}h_{k\ell} - h_{ik}h_{l\ell}\right)\pi^{k\ell}. \quad (91)$$

Now we can calculate

$$K^{ij}K_{ij} - K^2 = (h^{ik}h^{j\ell} - h^{ij}h^{k\ell})K_{ij}K_{k\ell} \quad (92)$$

$$= \frac{1}{h}\left(h^{ik}h^{j\ell} - h^{ij}h^{k\ell}\right)\left(\frac{1}{2}h_{ij}h_{mn} - h_{im}h_{nj}\right)\left(\frac{1}{2}h_{k\ell}h_{pq} - h_{kp}h_{q\ell}\right)\pi^{mn}\pi^{pq} \quad (93)$$

$$= \frac{\pi^{mn}\pi^{pq}}{h}\left(h^{ik}h^{j\ell} - h^{ij}h^{k\ell}\right)\left(\frac{1}{4}h_{ij}h_{mn}h_{k\ell}h_{pq} - \frac{1}{2}h_{ij}h_{mn}h_{kp}h_{q\ell} - \frac{1}{2}h_{im}h_{nj}h_{k\ell}h_{pq} + h_{im}h_{nj}h_{kp} \quad (94)$$

$$= \frac{1}{h}\left(h_{mp}h_{nq} - \frac{1}{2}h_{mn}h_{pq}\right)\pi^{mn}\pi^{pq}. \quad (95)$$

Now we want to rewrite the first term in the last line

$$h_{mp}h_{nq}\pi^{mn}\pi^{pq} = \frac{1}{2}(h_{mp}h_{nq}\pi^{mn}\pi^{pq} + h_{mp}h_{nq}\pi^{mn}\pi^{pq}) \quad (96)$$

$$= \frac{1}{2}(h_{mp}h_{nq}\pi^{mn}\pi^{pq} + h_{mq}h_{np}\pi^{mn}\pi^{qp}) \quad (97)$$

$$= \frac{1}{2}(h_{mp}h_{nq}\pi^{mn}\pi^{pq} + h_{mq}h_{np}\pi^{mn}\pi^{pq}), \quad (98)$$

where in the second line we renamed the (p, q) indices and in the third line we used the fact that the conjugate momentum is symmetric in its indices. Plugging this back into our original equation gives us

$$K^{ij}K_{ij} - K^2 = \frac{1}{\sqrt{h}}G_{ijk\ell}\pi^{ij}\pi^{k\ell}, \quad (99)$$

where $G_{ijk\ell} = \frac{1}{2\sqrt{h}}(h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell})$ is the Wheeler-DeWitt metric. The Hamiltonian given by the Legendre transformation is then

$$\mathcal{H} = \dot{h}_{ij}\pi^{ij} - \mathcal{L} = \dot{h}_{ij}\pi^{ij} - N\sqrt{h}[R - 2\Lambda + K^{ij}K_{ij} - K^2] \tag{100}$$

$$= \dot{h}_{ij}\pi^{ij} - N[G_{ijk\ell}\pi^{ij}\pi^{k\ell} + \sqrt{h}(R - 2\Lambda)]. \tag{101}$$

Plugging this back into the action gives

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^3x\,\mathrm{d}t\left[\dot{h}_{ij}\pi^{ij} - N\mathcal{H}_{DW}\right], \tag{102}$$

where $\mathcal{H}_{DW} = G_{ijk\ell}\pi^{ij}\pi^{k\ell} + \sqrt{h}(R - 2\Lambda)$.