Graviton-Graviton Scatter

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In order to compute the scattering amplitude of Mimetic Gravity, we must first know what the amplitude $\mathcal{A}$ is within General Relativity. Since we're only interested the scattering at the vertex, we only need to go to cubic order in the metric perturbation. Fortunately, we have Mathematica that is able to do all the hard work in deducing what that Lagrangian is. We can also streamline the calculation quite a bit by taking into account that the polarization tensor will be traceless and transverse i.e.

$$p^\mu \epsilon_{\mu\nu} = \epsilon^\mu{}_\mu = 0. \tag{1}$$

With all of that in mind (and doing an additional reduction via the action) we can write down the cubic order Einstein-Hilbert Lagrangian

$$\mathcal{L}^{(3)}_{EH} = -\frac{3}{2} h^{\mu\nu} h^{\lambda\rho} \partial_\lambda \partial_\rho h_{\mu\nu} - 9 h^{\mu\nu} \partial^\rho h^\lambda{}_\nu \partial_\rho h_{\mu\lambda} - 3 \partial^\lambda h^{\mu\nu} \partial^\rho h_{\mu\lambda} h_{\nu\rho}. \tag{2}$$

For simplicity we scale the Lagrangian in such a way that we eliminate the common factor of three. Now we proceed with differentiating with respect to the metric perturbation. The first derivative gives us

$$\begin{aligned} \frac{\delta \mathcal{L}^{(3)}_{EH}}{\delta h^{\mu\nu}} = &-\frac{1}{2}[\epsilon^1_{\mu\nu} h^{\lambda\rho} \partial_\lambda \partial_\rho h^{\mu\nu} + h^{\mu\nu} \epsilon_1^{\lambda\rho} \partial_\lambda \partial_\rho h_{\mu\nu} + h_{\mu\nu} h^{\lambda\rho} (-ip^1_\lambda)(-ip^1_\rho) \epsilon_1^{\mu\nu}] \\ &- 3[\epsilon_1^{\mu\nu} \partial^\rho h^\lambda{}_\nu \partial_\rho h_{\mu\lambda} + h^{\mu\nu} (-ip_1^\rho) \epsilon^\lambda{}_{1\nu} \partial_\rho h_{\mu\lambda} + h^{\mu\nu} \partial^\rho h^\lambda{}_\nu (-ip^1_\rho) \epsilon^1_{\mu\lambda}] \\ &- [(-ip_1^\lambda) \epsilon_1^{\mu\nu} h_{\nu\rho} \partial^\rho h_{\mu\lambda} + \partial^\lambda h^{\mu\nu} (-ip_1^\rho) \epsilon^1_{\mu\lambda} h_{\nu\rho} + \partial^\lambda h^{\mu\nu} \partial^\rho h_{\mu\lambda} \epsilon^1_{\nu\rho}], \end{aligned} \tag{3}$$

where we have used the Feynman rules $\partial_\mu \to (-ip_\mu)$ and $\frac{\delta}{\delta h^{\mu\nu}} \to \epsilon_{\mu\nu}$ and the numerical indices indicate which momenta on the Feynman diagram it represents. Since there are still h's left we must proceed with taking derivatives until we've eliminated all h's. The second derivative is

$$\begin{aligned}\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta \mathcal{L}^{(3)}_{EH}}{\delta h^{\mu\nu}} =& -\frac{1}{2}[\epsilon^1_{\mu\nu}(\epsilon_2^{\lambda\rho}\partial_\lambda\partial_\rho h^{\mu\nu} + h^{\lambda\rho}(-ip^2_\lambda)(-ip^2_\rho)\epsilon_2^{\mu\nu}) + \epsilon_1^{\lambda\nu}(\epsilon_2^{\mu\nu}\partial_\lambda\partial_\rho h_{\mu\nu} + h^{\mu\nu}(-ip^2_\lambda)(-ip^2_\rho)\epsilon^2_{\mu\nu}) \\ &+ (-ip^1_\lambda)(-ip^1_\rho)\epsilon_1^{\mu\nu}(\epsilon^2_{\mu\nu}h^{\lambda\rho} + h_{\mu\nu}\epsilon_2^{\lambda\rho})] \\ &- 3[\epsilon_1^{\mu\nu}((-ip^\rho_2)\epsilon^\lambda{}_{2\nu}\partial_\rho h_{\mu\lambda} + \partial^\rho h^\lambda{}_\nu(-ip^2_\rho)\epsilon^2_{\mu\lambda}) + (-ip^\rho_1)\epsilon^\lambda{}_{1\nu}(\epsilon_2^{\mu\nu}\partial_\rho h_{\mu\lambda} + h^{\mu\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) \\ &+ (-ip^1_\rho)\epsilon^1_{\mu\lambda}(\epsilon_2^{\mu\nu}\partial_\rho h_{\mu\lambda} + h^{\mu\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda})] \\ &- [(-ip^\lambda_1)\epsilon_1^{\mu\nu}(\epsilon^2_{\nu\rho}\partial^\rho h_{\mu\lambda} + h_{\nu\rho}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) + (-ip^\rho_1)\epsilon^1_{\mu\lambda}((-ip^\lambda_2)\epsilon_2^{\mu\nu}h_{\nu\rho} + \partial^\lambda h^{\mu\nu}\epsilon^2_{\nu\rho}) \\ &+ \epsilon^1_{\nu\rho}((-ip^\lambda_2)\epsilon_2^{\mu\nu}\partial^\rho h_{\mu\lambda} + \partial^\lambda h^{\mu\nu}\epsilon^2_{\mu\lambda})].\end{aligned} \tag{4}$$

Whew, we're almost done. There's only one more h left which means we can stop after the third (functional) derivative. Proceeding accordingly, we have

$$\begin{aligned}\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta}{\delta h^{\mu\nu}}\frac{\delta \mathcal{L}^{(3)}_{EH}}{\delta h^{\mu\nu}} =& -\frac{1}{2}[\epsilon^1_{\mu\nu}(\epsilon_2^{\lambda\rho}(-ip^3_\lambda)(-ip^3_\rho)\epsilon_3^{\mu\nu} + \epsilon_3^{\lambda\rho}(-ip^2_\lambda)(-ip^2_\rho)\epsilon_2^{\mu\nu}) \\ &+ \epsilon_1^{\lambda\nu}(\epsilon_2^{\mu\nu}(-ip^3_\lambda)(-ip^3_\rho)\epsilon^3_{\mu\nu} + \epsilon_3^{\mu\nu}(-ip^2_\lambda)(-ip^2_\rho)\epsilon^2_{\mu\nu}) \\ &+ (-ip^1_\lambda)(-ip^1_\rho)\epsilon_1^{\mu\nu}(\epsilon^2_{\mu\nu}\epsilon_3^{\lambda\rho} + \epsilon^3_{\mu\nu}\epsilon_2^{\lambda\rho})] \\ &- 3[\epsilon_1^{\mu\nu}((-ip^\rho_2)\epsilon^\lambda{}_{2\nu}(-ip^3_\rho)\epsilon^3_{\mu\lambda} + (-ip^\rho_3)\epsilon^\lambda{}_{3\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) \\ &+ (-ip^\rho_1)\epsilon^\lambda{}_{1\nu}(\epsilon_2^{\mu\nu}(-ip^3_\rho)\epsilon^3_{\mu\lambda} + \epsilon_3^{\mu\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) \\ &+ (-ip^1_\rho)\epsilon^1_{\mu\lambda}(\epsilon_2^{\mu\nu}(-ip^3_\rho)\epsilon^3_{\mu\lambda} + \epsilon_3^{\mu\nu}(-ip^2_\rho)\epsilon^2_{\mu\lambda})] \\ &- [(-ip^\lambda_1)\epsilon_1^{\mu\nu}(\epsilon^2_{\nu\rho}(-ip^\rho_3)\epsilon^3_{\mu\lambda} + \epsilon^3_{\nu\rho}(-ip^2_\rho)\epsilon^2_{\mu\lambda}) \\ &+ (-ip^\rho_1)\epsilon^1_{\mu\lambda}((-ip^\lambda_2)\epsilon_2^{\mu\nu}\epsilon^3_{\nu\rho} + (-ip^\lambda_3)\epsilon_3^{\mu\nu}\epsilon^2_{\nu\rho}) \\ &+ \epsilon^1_{\nu\rho}((-ip^\lambda_2)\epsilon_2^{\mu\nu}(-ip^\rho_3)\epsilon^3_{\mu\lambda} + (-ip^\lambda_3)\epsilon_3^{\mu\nu}\epsilon^2_{\mu\lambda})].\end{aligned} \tag{5}$$

There are a few ways we can simplify the above mess. First we recognize that each term in the Lagrangian has a factor of $(-i)^2 = -1$ which calls all of the minus signs. Next we take advantage of the fact that in four dimensions we can write $\epsilon_{\mu\nu} = \epsilon_\mu \epsilon_\nu$. With these two observations in mind, we can simplify the above expression immensely as being

$$\begin{aligned} \frac{\delta}{\delta h^{\mu\nu}} \frac{\delta}{\delta h^{\mu\nu}} \frac{\delta \mathcal{L}_{EH}^{(3)}}{\delta h^{\mu\nu}} &= \frac{1}{2}(\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot p_1 + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot p_2 + \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot p_3)^2 \\ &+ \frac{1}{2}(\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot p_2 + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot p_3 + \epsilon_3 \cdot \epsilon_1 \epsilon_2 \cdot p_1)^2 \\ &+ 6(\epsilon_1 \cdot \epsilon_2)(\epsilon_2 \cdot \epsilon_3)(\epsilon_3 \cdot \epsilon_1)(p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1). \end{aligned} \tag{6}$$

Now that our expression is much cleaner, it is far easier to see there are a few more simplifications that we can do. For example, from energy/momentum conservation, we have

$$p_1 + p_2 + p_3 = 0 \Leftrightarrow p_1^2 + p_2^2 + p_3^2 + 2p_1 \cdot p_2 + 2p_2 \cdot p_3 + 2p_3 \cdot p_1 = 0 \tag{7}$$

$$\Rightarrow p_1 \cdot p_2 + p_2 \cdot p_3 + p_3 \cdot p_1 = 0, \qquad p_i^2 = -m_i^2 = 0 \ \ \forall i \in \{1, 2, 3\}. \tag{8}$$

So we can safely discard the last term in our expression. Lastly, from momentum conservation we can see that $p_3 = -(p_1 + p_2)$ (and likewise $p_2 = -(p_3 + p_1)$ and $p_1 = -(p_2 + p_3)$). Plugging this and the fact that the polarization tensor is transverse we find that the two expressions inside the parentheses are the same up to a minus sign. But taking the power of two into account, the terms are exactly the same. So we finally get

$$\frac{\delta}{\delta h^{\mu\nu}} \frac{\delta}{\delta h^{\mu\nu}} \frac{\delta \mathcal{L}_{EH}^{(3)}}{\delta h^{\mu\nu}} = [(\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_1) + (\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_2) + (\epsilon_3 \cdot \epsilon_1)(\epsilon_2 \cdot p_3)]^2. \tag{9}$$