Electricity & Magnetism

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Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \varepsilon_0 = \mu_0 = 1$. The d'Alembert and Laplace operators are defined to be $\Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \nabla^2$ and $\nabla^2 = \partial_i \partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors.

Here we gather all of the field theoretic work on E&M. This document is organized as follows: first we count the degrees of freedom of massless E&M, then massive E&M (also known as Proca theory), then we derive the force law for both Lagrangians.

1 Massless Degrees of Freedom

We first start with the Lagrangian for electromagnetism in flat space

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1.1}$$

where $F_{\mu\nu}$ is the field strength tensor for electromagnetism and $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$. Next we write $F_{\mu\nu}$ in terms of the 4-potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.2}$$

where A_μ is the 4-potential with $\mu = 0, \ldots, 3$. Now we express the Lagrangian purely in terms of the 4-potential

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \tag{1.3}$$
$$= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu. \tag{1.4}$$

We can simplify this expression further by first writing the 4-potential in terms of its constituent field components, the electric and magnetic potential

$$A_\mu = (A_0, A_i), \tag{1.5}$$

where A_0 is the electric potential and A_i is the magnetic potential. Next we can perform a 3+1 decomposition to the Lagrangian and get the following expression

$$\mathcal{L} = \frac{1}{2}(\partial_0 A_i - \partial_i A_0)(\partial_0 A^i - \partial^i A_0) - \frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i) \tag{1.6}$$
$$= \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i). \tag{1.7}$$

We are now ready to plug this Lagrangian into the action, S. This yields

$$S = \int \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i)\, \mathrm{d}^4x\,, \tag{1.8}$$

where we're integrating over 3-spatial dimensions and one time dimension. Next we integrate by parts on the second term to get the following

$$S = \int \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 + \frac{1}{2}A_j(\nabla^2 A^j - \partial^j \partial_i A^i)\, \mathrm{d}^4x\,, \tag{1.9}$$

where we implicitly assume that A_j goes to zero at infinity. To simplify things even further, we make use of a theorem from linear algebra, where we can write any vector field as a longitudinal and transverse part

$$A_i = A_i^T + \partial_i \alpha, \quad \partial^i A_i^T = 0. \tag{1.10}$$

Replacing the A_i with it's Helmholtz decomposition in our action gives us

$$S = \int \frac{1}{2}(\dot{A}_i^T + \partial_i \dot{\alpha} - \partial_i A_0)^2 + \frac{1}{2}A_i^T \nabla^2 A_T^i + \frac{1}{2}\partial_i \alpha \nabla^2 A_T^i\, \mathrm{d}^4x\,. \tag{1.11}$$

Using integration by parts on the last term while assuming that the scalar function α goes to zero at infinity, we find that the last term is identically zero. This brings our action to take the form

$$S = \int \frac{1}{2}(\dot{A}_i^T + \partial_i \dot{\alpha} - \partial_i A_0)^2 + \frac{1}{2} A_i^T \nabla^2 A_T^i \, \mathrm{d}^4 x \,. \tag{1.12}$$

Now we are ready to count the degrees of freedom for flat space E+M. First we note the following gauge transformation law

$$A_\mu \to A_\mu + \partial_\mu \Lambda. \tag{1.13}$$

This implies that

$$A_0 \to A_0 + \dot{\Lambda}, \quad A_i^T \to A_i^T, \quad \alpha \to \Lambda. \tag{1.14}$$

Looking at the action, we can conclude that A_0 is an auxiliary field. And thus, we can eliminate it using the equations of motion we derived in the previous expression.

$$\frac{\delta \mathcal{L}}{\delta A_0} = \nabla^2(\dot{\alpha} - A_0) = 0 \Rightarrow A_0 = \dot{\alpha}. \tag{1.15}$$

The last step we justify by invoking the fact that the kernel of a linear operator is just the zero vector. With this fact in mind, our action reduces down to

$$S = \frac{1}{2} \int A_i^T \Box A_T^i \, \mathrm{d}^4 x \,. \tag{1.16}$$

From here it is obvious to see that the action under this Lagrangian has only two degrees of freedom as opposed to the 4 by introducing the 4-potential. The condition that $\partial_i A_T^i = 0$ constrains the action to at most 3 degrees of freedom. However, by choosing the gauge $\delta A_\mu = \partial_\mu \Lambda$ and eliminating the pure gauge part α, we reduced the degrees of freedom to just two.

2 Massive Degrees of Freedom

We start with the Lagrangian for massive electromagnetism in flat space

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu, \tag{2.1}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor, $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$, m is the mass of the photon, and A_μ is the 4-potential. Next we write $F_{\mu\nu}$ in terms of the 4-potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.2}$$

Next we write the 4-potential as

$$A_\mu = (A_0, A_i). \tag{2.3}$$

And now from the massless photon calculation, we can jump straight to the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i) - \frac{1}{2}m^2 A_0 A^0 - \frac{1}{2}m^2 A_i A^i. \tag{2.4}$$

Plugging the Lagrangian into the action S gives us

$$S = \int \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i) - \frac{1}{2}m^2 A_0 A^0 - \frac{1}{2}m^2 A_i A^i \,\mathrm{d}^4x\,. \tag{2.5}$$

Taking advantage of the result we calculated in the case of the massless photon while integrating certain terms out we get

$$S = \int \frac{1}{2}(\dot{A}_i^T + \partial_i\dot{\alpha} - \partial_i A_0)^2 + \frac{1}{2}A_T^i \nabla^2 A_i^T + \frac{1}{2}m^2 A_0^2 - \frac{1}{2}m^2(A_i^T + \partial_i\alpha)^2 \,\mathrm{d}^4x\,. \tag{2.6}$$

From (6) we observe that there are no $\dot{A}$ terms and thus we conclude A_0 is an auxiliary field. Meaning we can use it's equations of motions to eliminate it from the action. With this in mind, we're prepared to plug in the Lagrangian into the Euler-Lagrange equation

$$\frac{\delta\mathcal{L}}{\delta A_0} = \nabla^2(\dot{\alpha} - A_0) - m^2 A_0 = 0 \Rightarrow A_0 = D\dot{\alpha}, \quad D \equiv \frac{\nabla^2}{\nabla^2 + m^2}. \tag{2.7}$$

Plugging this equation into the action gives us

$$S = \frac{1}{2}\int \left[A_T^i(\Box - m^2)A_i^T - (\dot{\alpha}\nabla^2 + \nabla^2\dot{\alpha}D + \dot{\alpha}\nabla^2 D)\dot{\alpha} - m^2(D\dot{\alpha})^2 - m^2\partial_i\alpha\partial^i\alpha\right] \mathrm{d}^4x\,. \tag{2.8}$$

From here we can no longer identify anymore auxiliary fields and thus we conclude that for massive E&M, the 4-potential carries 3 degrees of freedom: two for A_i^T and one for α.

3 Coulomb's Law

We first start with the Lagrangian for electromagnetism in flat space coupled with a source

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + j^\mu A_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{3.1}$$

where $F_{\mu\nu}$ is the field strength tensor for E&M, A_μ is the 4-potential, and j_μ is the source current. And we are subject to the constraint that $\partial_\mu j^\mu = 0$. We can simplify this expression by breaking it down into the constituent components of the 4-potential and the source current by

$$A_\mu = (A_0, A_i), \quad j^\mu = (\rho, j^i), \tag{3.2}$$

where ρ and j_i are the charge and current density respectively. Combining the previous two expressions gets us

$$\mathcal{L} = \frac{1}{2}(\partial_0 A_i - \partial_i A_0)^2 - \frac{1}{2}\partial_i A_j(\partial^i A^j - \partial^j A^i) + \rho A_0 + j^i A_i. \tag{3.3}$$

From Helmholtz Theorem, we know we can write any 3-vector as the sum of a curl-less and divergence-less part. So we can write

$$A_i = A_i^T + \partial_i \alpha, \quad j_i = j_i^T + \partial_i \gamma, \tag{3.4}$$

where

$$\partial^i A_i^T = \partial^i j_i^T = 0. \tag{3.5}$$

Putting the previous expression into the action while taking advantage of the 3-potential and vector current's decomposition yields

$$S = \int \frac{1}{2}(\dot{A_i^T} + \partial_i\dot{\alpha} - \partial_i A_0)^2 + \frac{1}{2}A_T^i \nabla^2 A_i^T + \rho A_0 + j_T^i A_i^T + \partial_i\gamma\partial^i\alpha \, \mathrm{d}^4x \,. \tag{3.6}$$

Since our source, j_μ, is conserved the constraint equation becomes

$$\dot{\rho} + \nabla^2\gamma = 0 \Rightarrow \gamma = -\frac{1}{\nabla^2}\dot{\rho}. \tag{3.7}$$

Next we must eliminate all of the redundant fields within the Lagrangian. Since A_0 has no time derivatives, it is thus an auxiliary field and we can eliminate it using it's EOM. We get

$$\frac{\delta\mathcal{L}}{\delta A_0} = \rho + \nabla^2(\dot{\alpha} - A_0) = 0 \Rightarrow A_0 = \dot{\alpha} + \frac{1}{\nabla^2}\rho. \tag{3.8}$$

With the equation of motion in mind, the action reads

$$S = \int \frac{1}{2}A_T^i \Box A_i^T + j_T^i A_i^T + \frac{1}{2}\left(\frac{1}{\nabla^2}\partial_i\rho\right)^2 + \rho\dot{\alpha} + \rho\frac{1}{\nabla^2}\rho - \alpha\nabla^2\gamma \, \mathrm{d}^4x \,, \tag{3.9}$$

where we've integrated by parts on the first and last term. Once we plug in the equation of motion into the action we get

$$S = \int \frac{1}{2}A_T^i \Box A_i^T + j_T^i A_i^T + \frac{1}{2}\rho\frac{1}{\nabla^2}\rho \, \mathrm{d}^4x \,. \tag{3.10}$$

We are now ready to derive the force law of E and M. First we notice that the last term is the total electrostatic potential energy density. To get the force law, our best bet is through this piece. So we first write

$$U_E = \int \frac{1}{2}\rho\frac{1}{\nabla^2}\rho \, \mathrm{d}^3\mathbf{x} \tag{3.11}$$

$$= \frac{1}{2}\int \rho(t,\mathbf{x}) \int \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{p^2}\rho(t,\mathbf{p}) \, \mathrm{d}^3\mathbf{p} \, \mathrm{d}^3\mathbf{x} \,, \tag{3.12}$$

Next we note that

$$\rho(t,\mathbf{p}) = \int e^{-i\mathbf{p}\cdot\mathbf{x}}\rho(t,\mathbf{x}) \, \mathrm{d}^3\mathbf{x} \,, \tag{3.13}$$

which causes the expression in (3.12) to become

$$U_E = \frac{1}{2}\int\int\int \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{p^2}\rho(t,\mathbf{x})\rho(t,\mathbf{x}')\,\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{x}\,. \tag{3.14}$$

Using the formula

$$\frac{1}{(2\pi)^3}\int \frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}}{p^2}\,\mathrm{d}^3\mathbf{p} = \frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}, \tag{3.15}$$

expression (14) becomes

$$U_E = \frac{1}{8\pi}\int\int \frac{\rho(t,\mathbf{x})\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{x}\,. \tag{3.16}$$

Next we can consider this charge density as describing the electrostatic interactions of two point charges as such

$$\rho(t,\mathbf{x}) = Q\delta^3(\mathbf{x}) + q\delta^3(\mathbf{x}-\mathbf{r}). \tag{3.17}$$

Plugging this into the expression for U_E gives

$$\begin{aligned} U_E = \frac{1}{4\pi}\int\int \frac{(Qq\delta^3(\mathbf{x})\delta^3(\mathbf{x}'-\mathbf{r}))}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{x} + \frac{1}{4\pi}\int\int \frac{(Q^2\delta^3(\mathbf{x})\delta^3(\mathbf{x}'))}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{x} \\ +\frac{1}{4\pi}\int\int \frac{(q^2\delta^3(\mathbf{x}-\mathbf{r})\delta^3(\mathbf{x}'-\mathbf{r}))}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}^3\mathbf{x}'\,\mathrm{d}^3\mathbf{x}\,. \end{aligned} \tag{3.18}$$

If U_E represents the total energy of the system, then the last two terms would be the energy that the point charges gain when they interact with themselves. Since we only really care about what's going on between these two charges, we can omit these two terms in favor of the very first term. Once we carry out the integral for the very first term, we get

$$U_{E,\mathrm{int}} = \frac{Qq}{4\pi}\frac{1}{r}. \tag{3.19}$$

Differentiating with respect to r gives us the force

$$\mathbf{F} = \frac{Qq}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}. \tag{3.20}$$