Einstein-Cartan Theory

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Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_P = \frac{1}{\sqrt{8\pi G}} \approx 2.43 \times 10^{18}$ GeV. The d'Alembert and Laplace operators are defined to be $\Box = \partial_\mu \partial^\mu$ and $\nabla^2 = \partial_i \partial^i$ respectively. We use boldface letters $\mathbf{x}$ to indicate 3-vectors and we use x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll [1].

We are interested in formulating General Relativity in terms of a coordinate free basis. We do this with the hope of eventually working up to the Kodoma State which is written in terms of the coordinate-free action. We start off with a tangent space at a point p on a manifold M, $\mathrm{T}_p\mathrm{M}$. In differential geometry, we typically take our basis vectors to be partial derivatives. As a result, we pick for a basis for T_pM to be given by the partial derivatives with respect to the coordinates at that point $\hat{e}_{(\mu)} \equiv \partial_\mu$. Because the basis vectors for co-vectors are one forms, a basis for the cotangent space T_p^*M is given by the gradients of the coordinate functions, $\hat{\theta}^{(\mu)} = \mathrm{d}x^\mu$.

Let us imagine that at each point in the manifold we introduce a set of basis vectors $\hat{e}_{(a)}$ (we shall restrict ourselves to reference these coordinates with Latin indices as opposed to Greek indices which will denote coordinates). Let these basis vectors be "orthonormal". That is to say, if the canonical form of the metric is written η_{ab}, we demand that the inner product of our basis vectors (i.e. the metric) be

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab}. \tag{1}$$

We refer to the $\{\hat{e}_{(a)}, \hat{e}_{(b)}\}$ as the tetrad, or vielbein basis. We can express our old basis vectors $\hat{e}_{(\mu)} = \partial_\mu$ in terms of the new ones:

$$\hat{e}_{(\mu)} = e_\mu{}^a \hat{e}_{(a)}, \tag{2}$$

$e_\mu{}^a$ is an n×n invertible matrix (we'll call $e_\mu{}^a$ the vielbeins/tetrads) $e^\mu{}_a$ is the inverse so

$$e^\mu{}_a e_\nu{}^a = \delta^\mu{}_\nu, \qquad e_\mu{}^a e^\mu{}_b = \delta^a{}_b \Rightarrow \hat{e}_{(a)} = e^\mu{}_a \hat{e}_{(\mu)}. \tag{3}$$

Let's justify the second equality from the fact that $e^\mu{}_a e_\nu{}^a = \delta^\mu{}_\nu$

$$\big(\hat{e}_{(\mu)} = e_\mu{}^a \hat{e}_{(a)}\big) e^\mu{}_b \Rightarrow e^\mu{}_b \hat{e}_{(\mu)} = e^\mu{}_b e_\mu{}^a \hat{e}_{(a)} \tag{4}$$

$$= \delta^a_b \hat{e}_{(a)} \tag{5}$$

$$= \hat{e}_{(b)}. \tag{6}$$

We can see that

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = g(e^\mu{}_a \hat{e}_{(\mu)}, e^\nu{}_b \hat{e}_{(\nu)}) \tag{7}$$

$$= e^\mu{}_a e^\nu{}_b g(\hat{e}_{(\mu)}, \hat{e}_{(\nu)}) \tag{8}$$

$$= e^\mu{}_a e^\nu{}_b g(\partial_\mu, \partial_\nu) \tag{9}$$

$$= g_{\mu\nu} e^\mu{}_a e^\nu{}_b, \tag{10}$$

where in the second equality we made use of the fact that the metric is a bi-linear operator and hence $T(kv) = kT(v)\ \forall\ k \in \mathbb{F}$ and $v \in M$, and the last equality is the definition of the components of the metric tensor in the coordinate basis. We can construct an orthonormal basis of one-forms in T_pM, which we denote $\hat{\theta}^{(a)}$. They may be chosen to be compatible with the basis vectors, in the sense that

$$\hat{\theta}^{(\mu)} = e^\mu{}_a \hat{\theta}^{(a)}. \tag{11}$$

The orthonormality condition between the different basis vectors is

$$\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \hat{\theta}^{(a)}(e^{\mu}{}_{b}\hat{e}_{(\mu)}) \tag{12}$$

$$= e^{\mu}{}_{b}\hat{\theta}^{(a)}(\hat{e}_{(\mu)}) \tag{13}$$

$$= e^{\mu}{}_{b}e_{\nu}{}^{a}\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) \tag{14}$$

$$= e^{\mu}{}_{b}e_{\mu}{}^{a} \tag{15}$$

$$= \delta^{a}_{b}, \tag{16}$$

where we used the relation

$$\hat{\theta}^{(\nu)}(\hat{e}_{(\mu)}) = \mathrm{d}x^{\nu}\,(\partial_{\mu}) = \delta^{\nu}_{\mu}. \tag{17}$$

From all of the relations we've derived, we can express any arbitrary vector $V \in M$

$$V = V^{\mu}\partial_{\mu} = V^{\mu}\hat{e}_{(\mu)} = V^{\mu}e_{\mu}{}^{a}\hat{e}_{(a)} \equiv V^{a}\hat{e}_{(a)}, \tag{18}$$

so the tetrad $e_{\mu}{}^{a}$ can be identified as a change of basis matrix and allows us to switch between Latin indices (the tetrad basis) to Greek indices (the coordinate basis). We also see for a tensor that the vielbeins gives us

$$V^{a}_{\ \ b} = e_{\mu}{}^{a}V^{\mu}_{\ \ b} = e_{\mu}{}^{a}e^{\nu}{}_{b}V^{\mu}_{\ \ \nu}\,. \tag{19}$$

Apparently, it is also common to refer to the Greek indices as "curved" and the Latin indices as "flat", so we will also make use of that same language from time to time[1]. From everything we've learned so far, we can also write the inverse vielbeins in terms of the metric on spacetime as well as the Minkowski metric on the tangent space via the following:

$$g_{\mu\nu}e^{\mu}{}_{a}e^{\nu}{}_{b}e_{\lambda}{}^{a}e_{\rho}{}^{b} = \eta_{ab}e_{\lambda}{}^{a}e_{\rho}{}^{b} \tag{20}$$

$$g_{\lambda\rho} = \eta_{ab}e_{\lambda}{}^{a}e_{\rho}{}^{b} \tag{21}$$

$$\Rightarrow e^{\lambda}{}_{c}g_{\lambda\rho} = \eta_{ab}e_{\lambda}{}^{a}e_{\rho}{}^{b}e^{\lambda}{}_{c} \tag{22}$$

$$\Rightarrow e^{\lambda}{}_{c}g_{\lambda\rho} = \eta_{bc}e_{\rho}{}^{b} \tag{23}$$

$$\Rightarrow g_{\lambda\rho}e^{\lambda}{}_{c}\eta^{cd} = \delta^{d}_{b}e^{b}{}_{\rho}, \tag{24}$$

[1] When in Rome, do as the Romans do.

so we have

$$e_\mu{}^a = g_{\mu\nu} e^\nu{}_b \eta^{ab}, \qquad e^\mu{}_a = g^{\mu\nu} e_\nu{}^b \eta_{ab}. \tag{25}$$

We have introduced the vielbeins $e_\mu{}^a$ as components of a set of basis vectors, evaluated in a different basis. This is equivalent to thinking of the as the components of a (1, 1) tensor, $e = e_\mu{}^a \, \mathrm{d}x^\mu \, \hat{e}_{(a)}$. But this is a tensor we already know and love: the identity map.

$$\hat{e}_{(a)} \to \hat{e}_{(a')} = \Lambda^a{}_{a'} \hat{e}_{(a)}(x), \tag{26}$$

the $\Lambda(x)$'s are the only transformation rule that preserves the flat metric (with a Lorentzian signature) at each point:

$$\eta_{a'b'} = \Lambda^a{}_{a'} \Lambda^b{}_{b'} \eta_{ab}. \tag{27}$$

These transformations are therefore called local Lorentz transformations (LLTs). We also have the choice to alter our coordinates, which we'll call the general coordinate transformations (GCTs)

$$T^{a'\mu'}{}_{b'\nu'} = \Lambda^a{}_{a'} \frac{\partial x^{\mu'}}{\partial x^\mu} \Lambda^b{}_{b'} \frac{\partial x^\nu}{\partial x^{\nu'}} \eta_{ab} T^{a\mu}{}_{b\nu} \,. \tag{28}$$

Now let's compute the changes in the covariant derivative. Let $\omega_\mu{}^a{}_b$ be the spin connection and $X \in M$ be some vector in the manifold. We have

$$\nabla X \equiv (\nabla_\mu X^\nu) \, \mathrm{d}x^\mu \otimes \partial_\nu = (\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda) \, \mathrm{d}x^\mu \otimes \partial_\nu. \tag{29}$$

Next we find the same object in the mixed basis and then convert to the coordinate basis

$$
\begin{aligned}
\nabla X &= (\nabla_\mu X^a)\,\mathrm{d}x^\mu \otimes \hat{e}_{(a)} = \left(\partial_\mu X^a + \omega_\mu{}^a{}_b X^b\right)\mathrm{d}x^\mu \otimes \hat{e}_{(a)} && (30)\\
&= \left(\partial_\mu (X^\nu e_\nu{}^a) + \omega_\mu{}^a{}_b X^b\right)\mathrm{d}x^\mu \otimes \hat{e}_{(a)} && (31)\\
&= \left[e_\nu{}^a \partial_\mu X^\nu + X^\nu \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b X^b\right]\mathrm{d}x^\mu \otimes e^\lambda{}_a \hat{e}_{(\lambda)} && (32)\\
&= \left[\partial_\mu X^\lambda + e^\lambda{}_a X^\nu \partial_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e^\lambda{}_a e_\rho{}^b X^\rho\right]\mathrm{d}x^\mu \otimes \partial_\lambda && (33)\\
&= \left[\partial_\mu X^\nu + e^\nu{}_a X^\lambda \partial_\mu e_\lambda{}^a + \omega_\mu{}^a{}_b e^\nu{}_a e_\lambda{}^b X^\lambda\right]\mathrm{d}x^\mu \otimes \partial_\nu = (\partial_\mu X^\nu + \Gamma^\nu_{\mu\lambda} X^\lambda)\,\mathrm{d}x^\mu \otimes \partial_\nu. && (34)
\end{aligned}
$$

Thus, we can express the Christoffel symbols in terms of the tetrads and spin connection as

$$
\Gamma^\nu_{\mu\lambda} = e^\nu{}_a \partial_\mu e_\lambda{}^a + e^\nu{}_a e_\lambda{}^b \omega_\mu{}^a{}_b. \tag{35}
$$

Lets solve for the spin connection by applying the inverse vielbeins $e_\nu{}^c e^\lambda{}_d$ on the above equation to get

$$
\omega_\mu{}^c{}_d = e_\nu{}^c e^\lambda{}_d \Gamma^\nu_{\mu\lambda} - e^\lambda{}_d \partial_\mu e_\lambda{}^c, \tag{36}
$$

and similarly the derivative of the tetrad can be shown to be

$$
\partial_\mu e_\nu{}^a - e_\lambda{}^a \Gamma^\lambda_{\mu\nu} + e_\nu{}^b \omega_\mu{}^a{}_b = 0. \tag{37}
$$

The last equation can be recast as a constraint equation for the covariant derivative on the tetrad:

$$
\nabla_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0. \tag{38}
$$

This condition is called the tetrad postulate, or the absolute parallelism condition. Since both the Christoffel symbols as well as the spin connection are arbitrary, this equation is true regardless of the spin or Levi-Civita connection we use. Let us direct our attention toward the spin connection.

We've discussed how the tetrads and a general tensor transforms under GCTs and LLTs, but we have yet to discuss how the spin connection transforms. It is obvious to see that the spin connection will transform as a one form in its curved index (i.e. ${\omega_{\mu'}}^a{}_b = \frac{\partial x^\mu}{\partial x^{\mu'}} {\omega_\mu}^a{}_b$ this justifies us referring to the spin connection as a connection one-form), but how does it transform in its flat indices? The spin connection, like all gauge fields, transforms like

$$ {\omega_{\mu'}}^{a'}{}_{b'} = {\omega_\mu}^a{}_b \Lambda^{a'}{}_a \Lambda^b{}_{b'} - \Lambda^c{}_{b'} \partial_\mu \Lambda^{a'}{}_c. \tag{39} $$

We have previously said that we can think of objects like ${X_\mu}^a$ as mixed (1,1) tensors. It can also be useful to think of these objects as *Lie algebra*-, or *vector*-valued objects (takes in vectors/elements in the Lie algebra as its inputs and spits out scalars as its output). This interpretation lends us to think about acting the exterior derivative d (because these derivatives are defined for general p-forms) on ${X_\mu}^a$ to get Lie algebra-valued two-forms

$$ (\mathrm{d}X)^a_{\mu\nu} = \partial_\mu {X_\nu}^a - \partial_\nu {X_\mu}^a. \tag{40} $$

Because ${X_\mu}^a$ can already be thought of as a one form, acting the exterior derivative sends it to the space of two forms and it will transform appropriately. However, there is more work to be done on its flat indices. Since the transformation matrix will, in general, depend on the local coordinates which induces an inhomogeneous term in the transformation rule. With that in mind and our knowledge of how the spin connection transforms, we can formally introduce the covariant derivative on the internal flat space:

$$ \begin{aligned} (\mathcal{D}X)^a_{\mu\nu} &= (\mathrm{d}X)^a_{\mu\nu} + (\omega \wedge X)^a_{\mu\nu} &(41)\\ &= \partial_\mu {X_\nu}^a - \partial_\nu {X_\mu}^a + {\omega_\mu}^a{}_b {X_\nu}^b - {\omega_\nu}^a{}_b {X_\mu}^b. &(42) \end{aligned} $$

This formalism provides us with the opportunity to write down all of our quantities with the curved indices suppressed (which should make for a much less cumbersome notation!), providing us with very eloquent definitions for the torsion and curvature tensors. Writing

$$ e^a \equiv {e_\mu}^a \, \mathrm{d}x^\mu \,, \quad \omega^a{}_b \equiv {\omega_\mu}^a{}_b \, \mathrm{d}x^\mu \,, \tag{43} $$

we can define

$$T^a \equiv \mathcal{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b, \qquad R^a{}_b \equiv \mathcal{D}\omega^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{44}$$

These constructions are called the Cartan structure equations. We can show that this new definition of the torsion tensor is equivalent to the one given in GR by noticing

$$\Gamma^\lambda_{\mu\nu} = e^\lambda{}_a \partial_\mu e_\nu{}^a + e^\lambda{}_a e_\nu{}^b \omega_\mu{}^a{}_b, \tag{45}$$

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} &= e^\lambda{}_a \partial_\mu e_\nu{}^a - e^\lambda{}_a \partial_\nu e_\mu{}^a + e^\lambda{}_a e_\nu{}^b \omega_\mu{}^a{}_b - e^\lambda{}_a e_\mu{}^b \omega_\nu{}^a{}_b && (46)\\
&= e^\lambda{}_a (\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + e_\nu{}^b \omega_\mu{}^a{}_b - e_\mu{}^b \omega_\nu{}^a{}_b) && (47)\\
&= e^\lambda{}_a T_{\mu\nu}{}^a = T_{\mu\nu}{}^\lambda. && (48)
\end{aligned}$$

Our definitions for the torsion and curvature can be put to some good use. For example, we can see that

$$\begin{aligned}
\mathcal{D}\mathcal{D}V^a &= \mathcal{D}(\mathrm{d}V^a) + \mathcal{D}(\omega^a{}_b \wedge V^b) && (49)\\
&= \omega^a{}_b \wedge \mathrm{d}V^b + \mathrm{d}(\omega^a{}_b \wedge V^b) + \omega^a{}_c \wedge \omega^c{}_b \wedge V^b && (50)\\
&= (\mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b) \wedge V^b = R^a{}_b \wedge V^b, && (51)
\end{aligned}$$

where we used the conditions $\mathrm{d}(\mathrm{d}V^a) = 0$ and $\mathrm{d}(\alpha_p \wedge \beta_q) = \mathrm{d}\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge \mathrm{d}\beta_q$, where α_p is a p-form and β_q is a q-form. When we parallel transport these geometric objects through the internal flat space, we find

$$\begin{aligned}
\mathcal{D}T^a = \mathrm{d}T^a + \omega^a{}_b \wedge T^b &= \mathrm{d}\big(\mathrm{d}e^a + \omega^a{}_b \wedge e^b\big) + \omega^a{}_b \wedge (\mathrm{d}e^b + \omega^b{}_c \wedge e^c) && (52)\\
&= \mathrm{d}\omega^a{}_b \wedge e^b + \mathrm{d}\omega^a{}_b \wedge \omega^a{}_c \wedge e^b = R^a{}_b \wedge e^b, && (53)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}R^a{}_b &= \mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b && (54)\\
&= \mathrm{d}\omega^a{}_c \wedge \omega^c{}_b - \omega^a{}_c \wedge \mathrm{d}\omega^c{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_d && (55)\\
&= \omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_b + \omega^a{}_d \wedge \omega^d{}_c \wedge \omega^c{}_b && (56)\\
&= 0. && (57)
\end{aligned}$$

These last two relations are generalizations of the first and second Bianchi identities:

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu} = 0, \tag{58}$$

$$\nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\lambda} + \nabla_\nu R_{\alpha\beta\lambda\mu} = 0, \tag{59}$$

respectively. Lastly, we enforce a metric compatibility to our covariant derivative on the Minkowski metric:

$$\mathcal{D}_\mu \eta_{ab} = -\omega_\mu{}^c{}_a \eta_{cb} - \omega_\mu{}^c{}_b \eta_{ac} \tag{60}$$

$$= -\omega_{\mu ba} - \omega_{\mu ab} \tag{61}$$

$$= 0 \Rightarrow \omega_{\mu ba} = -\omega_{\mu ab}. \tag{62}$$

We are ready to throw these objects into the coordinate-free Palantini action

$$S_P[e,\omega] = \frac{1}{2\kappa^2}\int_M \frac{1}{2}\epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}, \tag{63}$$

where ϵ_{abcd} are the components of the totally anti-symmetric Levi-Civita symbol and $\epsilon_{0123} = -\epsilon^{0123} = 1$ and $\kappa^2 = 8\pi G$. Now we can vary the action with respect to our dynamical fields e^a and $\omega^a{}_b$. First we try the spin connection:

$$\delta_\omega S_P = \frac{1}{2\kappa^2}\int_M \frac{1}{2}\epsilon_{abcd} e^a \wedge e^b \wedge \delta_\omega R^{cd}. \tag{64}$$

Since our dynamics for the spin connection only enter through the curvature two-form, let us compute its variation with respect to the connection

$$\delta_\omega R^{cd} = \mathrm{d}\delta\omega^{cd} + \delta\omega^c{}_f \wedge \omega^{fd} + \omega^c{}_f \wedge \delta\omega^{fd} \tag{65}$$

$$= \mathrm{d}\delta\omega^{cd} + \omega^d{}_f \wedge \delta\omega^{cf} + \omega^c{}_f \wedge \delta\omega^{fd} \tag{66}$$

$$= \mathcal{D}\delta\omega^{cd}. \tag{67}$$

When we put this term back into the action, after an integration by parts, we get

$$\delta_\omega S_P = -\frac{1}{2\kappa^2}\int_M \frac{1}{2}\epsilon_{abcd}\big[\mathcal{D}\big(e^a \wedge e^b\big)\big] \wedge \delta\omega^{cd}. \tag{68}$$

Using the anti-symmetrization properties of the wedge product and the Levi-Civita symbol, we get

$$\delta_\omega S_P = -\frac{1}{2\kappa^2}\int_M \epsilon_{abcd}\mathcal{D}e^a \wedge e^b \wedge \delta\omega^{cd} = \frac{1}{2\kappa^2}\int_M \epsilon_{abcd}T^a \wedge e^b \wedge \delta\omega^{cd}. \tag{69}$$

And when we set the functional derivative equal to zero, we get

$$\frac{\delta S_P}{\delta\omega^{cd}} = \epsilon_{abcd}T^a \wedge e^b = 0. \tag{70}$$

Since this equation has to be true for all a, b, c, d and the vielbeins can't be zero since they are invertible, this forces us to the conclusion that

$$T^a = \mathcal{D}e^a = 0 \Rightarrow \mathcal{D}_\mu e_\nu{}^a - \mathcal{D}_\nu e_\mu{}^a = 0. \tag{71}$$

So we get a torsion free condition just based off of the equations of motion for the spin connection. We can take this constraint a step further by writing the covariant derivative in terms of said connection

$$\mathcal{D}_\mu e_\nu{}^a = \nabla_\mu e_\nu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b = 0 \tag{72}$$

$$\Rightarrow \omega_\mu{}^a{}_b[e] = -e^\nu{}_b \nabla_\mu e_\nu{}^a, \tag{73}$$

which implies that the spin connection is *not* a dynamical field since we've been able to eliminate it using its own equations of motion. And now the tetrad equations of motion:

$$\delta_e S_P[e,\omega] = \frac{1}{2\kappa^2}\int_M \frac{1}{2}\epsilon_{abcd}\big(\delta e^a \wedge e^b \wedge R^{cd} + e^a \wedge \delta e^b \wedge R^{cd}\big) \tag{74}$$

$$= \frac{1}{2\kappa^2}\int_M \epsilon_{abcd}\delta e^a \wedge e^b \wedge R^{cd}, \tag{75}$$

where we used the anti-symmetrization properties of the Levi-Civita symbol and the wedge product yet again. Setting the variation equal to zero gives us

$$\frac{\delta S_P}{\delta e^a} = \epsilon_{abcd}e^b \wedge R^{cd} = 0. \tag{76}$$

Expressing this equation in the coordinate basis gives us

$$\epsilon_{abcd}e_{\nu}{}^{b}R^{cd}{}_{\lambda\rho}\epsilon^{\mu\nu\lambda\rho}=0. \tag{77}$$

Inserting the completeness relation for the tetrads gives us

$$0=\epsilon_{a'bc'd'}e_{\alpha}{}^{a'}e_{\nu}{}^{b}e_{\gamma}{}^{c'}e_{\delta}{}^{d'}e^{\alpha}{}_{a}e^{\gamma}{}_{c}e^{\delta}{}_{d}R^{cd}{}_{\lambda\rho}\epsilon^{\mu\nu\lambda\rho} \tag{78}$$

$$=e\epsilon_{\alpha\nu\gamma\delta}\epsilon^{\mu\nu\lambda\rho}e^{\alpha}{}_{a}e^{\gamma}{}_{c}e^{\delta}{}_{d}R^{cd}{}_{\lambda\rho}, \tag{79}$$

where $e=\det\left(e_{\mu}{}^{a}\right)$. Now we use

$$\epsilon_{\alpha\nu\gamma\delta}\epsilon^{\mu\nu\lambda\rho}=\delta^{\mu}_{\alpha}\delta^{\lambda}_{\gamma}\delta^{\rho}_{\delta}+\delta^{\rho}_{\alpha}\delta^{\mu}_{\gamma}\delta^{\lambda}_{\delta}+\delta^{\lambda}_{\alpha}\delta^{\rho}_{\gamma}\delta^{\mu}_{\delta}-\delta^{\mu}_{\alpha}\delta^{\lambda}_{\delta}\delta^{\rho}_{\gamma}-\delta^{\rho}_{\alpha}\delta^{\mu}_{\delta}\delta^{\lambda}_{\gamma}-\delta^{\lambda}_{\alpha}\delta^{\rho}_{\delta}\delta^{\mu}_{\gamma}, \tag{80}$$

which turns the equation of motion (after making use of the anti-symmetrization properties of the curvature 2-form) brings us

$$-4e\left(e^{\rho}{}_{a}e^{\mu}{}_{c}e^{\lambda}{}_{d}+\frac{1}{2}e^{\mu}{}_{a}e^{\lambda}{}_{c}e^{\rho}{}_{d}\right)R^{cd}{}_{\lambda\rho}=0. \tag{81}$$

Next we recognize $R^{cd}{}_{\lambda\rho}=e_{\alpha}{}^{c}e_{\beta}{}^{d}R^{\alpha\beta}{}_{\lambda\rho}$ and

$$\det(g_{\mu\nu})=\det\left(\eta_{ab}e_{\mu}{}^{a}e_{\nu}{}^{b}\right) \tag{82}$$

$$=\det(\eta_{ab})(\det\left(e_{\mu}{}^{a}\right))^{2} \tag{83}$$

$$=-e^{2}, \tag{84}$$

implying $e=\sqrt{-g}$. Plugging these relations into the equation of motion gives us

$$\sqrt{-g}\left(e^{\rho}{}_{a}R^{\mu\lambda}{}_{\lambda\rho}+\frac{1}{2}e^{\mu}{}_{a}R^{\lambda\rho}{}_{\lambda\rho}\right)=0. \tag{85}$$

Recognizing that the first term is $-R^{\mu}{}_{\rho}$ and the second term is exactly the Ricci scalar R, acting the tetrad $e^{\nu a}$ on the equation while simply the expression to

$$\sqrt{-g}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)=0, \tag{86}$$

where in the last line we used

$$g^{\mu\nu}=e^{\mu a}e^{\nu}{}_{a}=\eta_{ab}e^{\mu a}e^{\nu b}. \tag{87}$$

Appendices

A Scalar Coupling

Lets see what happens if we were to couple the original Palantini action with some scalar field $\phi(x)$

$$S_P[e,\omega,\phi] = \frac{1}{2\kappa^2}\int_M \frac{1}{2}\epsilon_{abcd}\phi e^a \wedge e^b \wedge R^{cd}. \tag{A.1}$$

Next we'll vary the action with respect to the spin connection

$$\delta_\omega S_P = \frac{1}{2\kappa^2}\int_M \frac{1}{2}\epsilon_{abcd}\phi e^a \wedge e^b \wedge \delta_\omega R^{cd} \tag{A.2}$$

$$= \frac{1}{2\kappa^2}\int_M \epsilon_{abcd}\left(\phi T^a \wedge e^b + \frac{1}{2}(\mathcal{D}\phi)e^a \wedge e^b\right) \wedge \delta\omega^{cd}. \tag{A.3}$$

Setting the variation equal to zero gives us

$$\frac{\delta S_P}{\delta \omega^{cd}} = \epsilon_{abcd}\left(\phi T^a + \frac{1}{2}(\mathcal{D}\phi)e^a\right) \wedge e^b = 0 \Rightarrow T^a = -\frac{1}{2}(\mathcal{D}\ln\phi)e^a. \tag{A.4}$$

Interestingly, the scalar coupling induces a non-zero torsion on the action. Remembering that the torsion is given by the covariant derivative of the tetrad field, we get

$$\mathcal{D}e^a = -\frac{1}{2}(\mathcal{D}\ln\phi)e^a, \tag{A.5}$$

which reduces finding the torsion to finding the eigenvectors of the covariant derivative with eigenvalue $\frac{1}{2}\mathcal{D}\ln\phi$ (at least in this coordinate-free notation). In operator language, we can construct a Green's function for our differential operator. The other equations of motion we get from varying the other fields are

$$\frac{\delta S_P}{\delta e^a} = \epsilon_{abcd}\phi e^b \wedge R^{cd} = 0 \quad \frac{\delta S_P}{\delta \phi} = \epsilon_{abcd}e^a \wedge e^b \wedge R^{cd} = 0. \tag{A.6}$$

The second equality shows that the field ϕ acts as a Lagrange multiplier. Interestingly, this equation is actually implied by the variation in the tetrad fields, which implies a redundancy in our description.

References

[1] Carroll, Sean Michael. *"Spacetime and Geometry an Introduction to General Relativity."* Harlow: Pearson Education, 2014.

[2] LAGRAA, Meriem Hadjer *"On the equivalence between the Einstein-Cartan theory and General Relativity in presence of fermions"*