Energy Momentum Tensor for Generic Scalar Field Coupling to Gravity

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1 Introduction

We want to derive the energy-momentum tensor for a scalar field, ϕ, from variations of the metric, $g_{\mu\nu}$. In a typical QFT course, one can derive the energy-momentum tensor by simply considering Noether's Theorem. The theorem states that every symmetry of a Lagrangian, $\mathcal{L}$, carries with it a conserved quantity i.e.

$$\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} \Rightarrow \delta\mathcal{L} = 0. \tag{1.1}$$

In flat space, the Lagrangian for a (free) scalar field is given simply by

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi), \tag{1.2}$$

where $V(\phi)$ contains all the mass and self-interacting terms of the scalar field. The above expression is invariant under the transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$ where ϵ^μ is a constant. Under this transformation, the Lagrangian varies by

$$\mathcal{L} \to -\frac{1}{2}\eta'^{\mu\nu}\frac{\partial\phi'}{\partial x'^\mu}\frac{\partial\phi'}{\partial x'^\nu} - V(\phi') = -\frac{1}{2}\eta^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} - V(\phi) = \mathcal{L}, \tag{1.3}$$

where we used the fact that $\phi'(x') = \phi(x)$, $\eta'_{\mu\nu} = \eta_{\mu\nu}$ and

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\mu}. \tag{1.4}$$

So the Lagrangian is invariant under constant spacetime translation. Therefore we can identify the conserved quantity. Now we again let $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$ and we get

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) + \frac{\partial\mathcal{L}}{\partial x^\mu}\epsilon^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right) + \frac{\partial\mathcal{L}}{\partial x^\mu}\epsilon^\mu = 0, \tag{1.5}$$

where we used the Euler-Lagrange Equations with $\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$. Now we take $\delta\phi(x) = -\frac{\partial\phi}{\partial x^\nu}\epsilon^\nu = -\frac{\partial\phi}{\partial x_\nu}\epsilon_\nu$ and the total change in the Lagrangian is then

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}\right]\epsilon_\nu \equiv \partial_\mu T^{\mu\nu}\epsilon_\nu = 0, \tag{1.6}$$

where we can define the energy-momentum tensor for the scalar field to be

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L}. \tag{1.7}$$

Unfortunately, the problem with this definition of the energy-momentum tensor is that (1) in general it's not symmetric and (2) its not gauge invariant. These both are exemplified in E&M where the energy momentum tensor is written as

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\lambda)}\partial^\nu A^\lambda - g^{\mu\nu}\mathcal{L}, \tag{1.8}$$

which is clearly not symmetric, let alone gauge invariant. Thus, we need a definition of the energy-momentum tensor that is symmetric (because the Einstein tensor is symmetric) as well as gauge invariant. The definition that does this is

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{1.9}$$

where S is the action of whatever matter fields that exist in the Lagrangian.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = k_B = 1$. The reduced four dimensional Planck mass is $M_{\rm Pl} = (8\pi G)^{-1/2} \approx 2.43 \times 10^{18}\,\text{GeV}$. The d'Alembert and Laplace operators are defined to be $\Box \equiv \partial_\mu\partial^\mu = -\partial_t^2 + \nabla^2$ and $\nabla^2 = \partial_i\partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

2 The Derivation

Here we derive the proper form of the stress-energy tensor given arbitrary coupling to the curvature scalar. First we reproduce the relevant expression

$$S_M = -\frac{1}{2}\int \sqrt{-g}\big[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + \xi R\phi^2\big]\,\mathrm{d}^4x\,. \tag{2.1}$$

Now we vary with respect to the metric. We write

$$\begin{aligned} \delta S_M = &-\frac{1}{2}\int \delta\sqrt{-g}\,\mathrm{d}^4x\,\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + \xi R\phi^2\right] \\ &-\frac{1}{2}\int \sqrt{-g}\,\mathrm{d}^4x\,\left[\delta g^{\mu\nu}\,\partial_\mu\phi\partial_\nu\phi + \xi\,\delta R\,\phi^2\right]. \end{aligned} \tag{2.2}$$

We've done previous derivations for what $\delta\sqrt{-g}$ and δR are, so we will just quote those results

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu}\,, \qquad \delta R = R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}\,, \tag{2.3}$$

where the variation in the Ricci scalar can be further expanded into

$$\delta R = R_{\mu\nu}\,\delta g^{\mu\nu} + (g^{\mu\nu}g^{\lambda\rho} - g^{\mu\lambda}g^{\nu\rho})\nabla_\lambda[\nabla_\mu\,\delta g_{\nu\rho} + \nabla_\nu\,\delta g_{\mu\rho} - \nabla_\rho\,\delta g_{\mu\nu}]. \tag{2.4}$$

We also needed the following identities in order to compute the above

$$\delta g_{\mu\nu} = -g_{\mu\lambda}g_{\nu\rho}\,\delta g^{\lambda\rho}\,, \qquad g^{\mu\nu}\,\delta g_{\mu\nu} = -g_{\mu\nu}\,\delta g^{\mu\nu}\,. \tag{2.5}$$

Next we can throw these terms into the action to get

$$\begin{aligned} \delta S_M = &-\frac{1}{2}\int\left(-\frac{1}{2}\sqrt{-g}g_{\mu\nu}\,\delta g^{\mu\nu}\right)\mathrm{d}^4x\,\left[g^{\lambda\rho}\partial_\lambda\phi\partial_\rho\phi + m^2\phi^2 + \xi R\phi^2\right] \\ &-\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^4x\,\left[\delta g^{\mu\nu}\,\nabla_\mu\phi\nabla_\nu\phi + \xi\big(\phi^2 R_{\mu\nu}\,\delta g^{\mu\nu} + \delta g^{\mu\nu}\,g_{\mu\nu}\Box\phi^2 - \delta g^{\mu\nu}\,\nabla_\mu\nabla_\nu\phi^2\big)\right]. \end{aligned} \tag{2.6}$$

We can write this under a single integral sign

$$\begin{aligned} \delta S_M = -\frac{1}{2}\int\sqrt{-g}\,\mathrm{d}^4x\,\delta g^{\mu\nu}\Bigg[&\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\lambda\rho}\nabla_\lambda\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 g_{\mu\nu} + \xi\big(g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2\big) \\ &+\xi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\phi^2\Bigg]. \end{aligned} \tag{2.7}$$

Now by boldly defining the stress-energy tensor by the previous formula we gave, we're left with

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\lambda\rho}\nabla_\lambda\phi\nabla_\rho\phi - \frac{1}{2}m^2\phi^2 g_{\mu\nu} + \xi\big(g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2\big) + \xi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\phi^2. \tag{2.8}$$

We can also compute the trace given by

$$T \equiv g^{\mu\nu}T_{\mu\nu} = -\nabla^\mu\phi\nabla_\mu\phi - 2m^2\phi^2 + 3\xi\Box\phi^2 - \xi R\phi^2. \tag{2.9}$$

We can do partial integration on the first gradient-squared term to get

$$T = -\nabla^\mu(\phi\nabla_\mu\phi) + \phi\Box\phi - m^2\phi^2 - \xi R\phi^2 - m^2\phi^2 + 3\xi\Box\phi^2 \tag{2.10}$$

$$= -\nabla^\mu(\phi\nabla_\mu\phi) - \phi\big(-\Box + m^2 + \xi R\big)\phi - m^2\phi^2 + 3\xi\nabla^\mu(2\phi\nabla_\mu\phi) \tag{2.11}$$

$$= -(1-6\xi)\nabla^\mu(\phi\nabla_\mu\phi) - m^2\phi^2, \tag{2.12}$$

where we made use of the Klein-Gordon equations of motion for the scalar field

$$-\Box\phi + m^2\phi + \xi R\phi = 0, \tag{2.13}$$

as well as the fact that

$$\Box\phi^2 = \nabla^\mu\nabla_\mu\phi^2 = 2\nabla^\mu(\phi\nabla_\mu\phi). \tag{2.14}$$

We can clearly see that for $\xi = 1/6$ and $m = 0$ that the trace vanishes. This tells us that the stress-energy tensor and hence the action is conformally invariant when we set $\xi = 1/6$ and $m = 0$. Finally, we can express the energy-momentum tensor in an alternative way show to be

$$\begin{aligned} T_{\mu\nu} = (1-2\xi)\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}(1-4\xi)g_{\mu\nu}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 g_{\mu\nu} + \xi G_{\mu\nu}\phi^2 \\ + 2\xi(\phi\Box\phi g_{\mu\nu} - \phi\partial_\mu\partial_\nu\phi + \phi\Gamma^\lambda_{\mu\nu}\partial_\lambda\phi), \end{aligned} \tag{2.15}$$

where $(\nabla\phi)^2 \equiv g^{\lambda\rho}\nabla_\lambda\phi\nabla_\rho\phi$.