

Feynman Rules for Cubic Scalar Field

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October 18, 2023

1 Cubic Field

Here we are interested in computing the Feynman rules for a Lagrangian that is cubic in a massless, free scalar field. The Lagrangian is given by the following

$$\mathcal{L}_3 = -\frac{\lambda}{3!}\phi^3. \quad (1.1)$$

To find the factor that will be placed at each vertex of our Feynmann diagram, we need to keep taking derivatives until the scalar field has completely vanished from the Lagrangian. We proceed with the first derivative

$$\frac{\delta\mathcal{L}_3}{\delta\phi} = -\frac{1}{2}\lambda\phi^2. \quad (1.2)$$

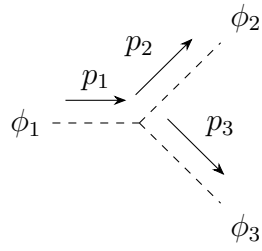
Since ϕ is still present within the Lagrangian we need to take another derivative. The second derivative is

$$\frac{\delta^2\mathcal{L}_3}{\delta\phi^2} = -\lambda\phi. \quad (1.3)$$

Now that ϕ appears linearly within the functional, we need only take one more derivative. Hence:

$$\frac{\delta^3\mathcal{L}_3}{\delta\phi^3} = -\lambda. \quad (1.4)$$

Next we can write down the amplitude for this Lagrangian.



Since this process has only one vertex, calculating the amplitude is fairly trivial. The amplitude is merely $\mathcal{A} = \lambda$. So that was easy, which is exactly what we expected since this is the simplest lowest order tree diagram that contributes to the overall scattering.

2 Quadratic Derivative

We are interested in calculating the scattering amplitude for a free and massless scalar field whose coupled to its own derivatives. The Lagrangian we wish to work with is

$$\mathcal{L}_3 = -\frac{\lambda}{3!}\phi(\partial\phi)^2, \quad (2.1)$$

where λ is the coupling constant of our toy theory, ϕ is the scalar field and $(\partial\phi)^2 = \partial_\mu\phi\partial^\mu\phi$ is a shorthand for the kinetic energy of the field. The first derivative is given by

$$\frac{\delta\mathcal{L}_3}{\delta\phi} = -\frac{\lambda}{3!}(\partial\phi)^2 - \frac{\lambda}{3}\phi(-ip_i^\mu)\partial_\mu\phi, \quad (2.2)$$

where we use the Feynman rule that

$$\frac{\delta\partial_\mu\phi\partial^\mu\phi}{\delta\phi} = 2(-ip^\mu)\partial_\mu\phi. \quad (2.3)$$

Taking the second derivative with respect to ϕ gives us

$$\frac{\delta^2\mathcal{L}_3}{\delta\phi^2} = -\frac{\lambda}{3}(ip_j^\mu)\partial_\mu\phi - \frac{\lambda}{3}(-ip_i^\mu)\partial_\mu\phi - \frac{\lambda}{3}\phi(-ip_{j,\mu})(-ip_i^\mu). \quad (2.4)$$

Differentiating one last time with respect to ϕ gets us,

$$\frac{\delta^3\mathcal{L}_3}{\delta\phi^3} = -\frac{1}{3}\lambda(ip_{k,\mu})(ip_j^\mu) - \frac{1}{3}\lambda(ip_{k,\mu})(ip_i^\mu) - \frac{1}{3}\lambda(ip_{j,\mu})(ip_i^\mu) \quad (2.5)$$

$$= \frac{\lambda}{3}(p_i \cdot p_j + p_j \cdot p_k + p_k \cdot p_i). \quad (2.6)$$

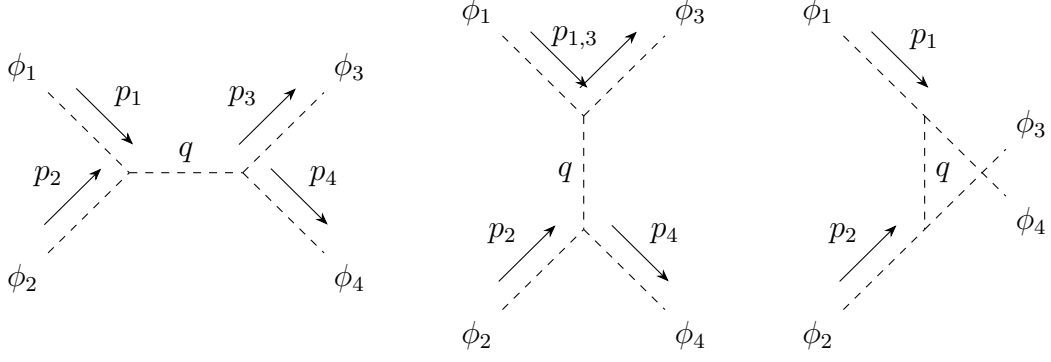
Before we carry on with the calculating, it is useful for us to work in Mandelstam variables (s, t, u) where

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 2p_1 \cdot p_2, \quad (2.7)$$

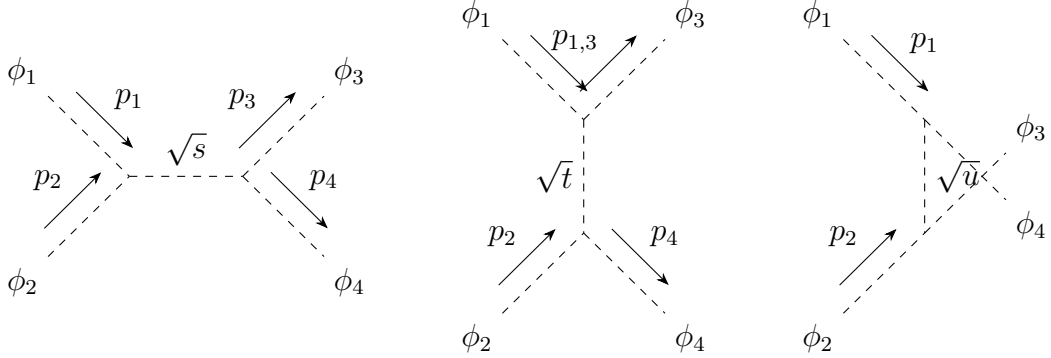
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = -2p_1 \cdot p_3, \quad (2.8)$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 = -2p_1 \cdot p_4, \quad (2.9)$$

where we used the fact that $m_1 = m_2 = m_3 = m_4 = 0$. Next we need to compute the vertex rule for each diagram in Mandelstam variables. We introduce the parameter $\alpha = \frac{1}{3}\lambda(p_i \cdot p_j + p_j \cdot p_k + p_k \cdot p_i)$ for convenience.



Now we can compute the scattering amplitude, \mathcal{A} , for this Lagrangian. Since there are three possible diagrams that can be constructed from this theory, it is typical for one to break up each amplitude contribution into channels: s , t , and u . The most left diagram is called the s -channel, the center diagram is called the t -channel, and the right most diagram is the u -channel. Griffiths lays out a prescription on how to calculate the momenta of the internal line, but we can skip to the right answer.



First we calculate the vertex rule for the s -channel. From the diagram we can see

$$p_i = p_1, \quad p_j = p_2, \quad p_k = -(p_1 + p_2) \Rightarrow \alpha_s = -\frac{\lambda}{6}s. \quad (2.10)$$

The t -channel momenta and vertex rule is given by

$$p_i = p_1, \quad p_j = -(p_1 - p_3), \quad p_k = -p_3 \Rightarrow \alpha_t = -\frac{\lambda}{6}t. \quad (2.11)$$

And finally, the u -channel is shown to be

$$p_i = p_1, \quad p_j = -p_4, \quad p_k = -(p_1 - p_4) \Rightarrow \alpha_u = -\frac{\lambda}{6}u. \quad (2.12)$$

However, there is a constraint on the Mandelstam variables in that they satisfy the following equation

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2, \quad (2.13)$$

where m_i is the mass of the i -th particle. Since we're assuming all the scattered particles are massless, we can eliminate one of the variables in favor of the other two. The convention is to write $u = -(s+t)$. Now we can calculate the amplitude for each diagram

$$\mathcal{A}_s = \frac{\alpha_s^2}{s}, \quad \mathcal{A}_t = \frac{\alpha_t^2}{t}, \quad \mathcal{A}_u = \frac{\alpha_u^2}{u}. \quad (2.14)$$

Thus the total scattering amplitude, the amplitude that is contributed by all three Feynman diagrams is

$$\mathcal{A} = \mathcal{A}_s + \mathcal{A}_t + \mathcal{A}_u. \quad (2.15)$$

Remembering that $\alpha_p = \alpha_p(s, t, u)$, we can simplify the total amplitude just in terms of the coupling constant and Mandelstam variables. Proceeding accordingly, we get

$$\mathcal{A} = \frac{1}{s} \left(\frac{\lambda}{6} \right)^2 s^2 + \frac{1}{t} \left(\frac{\lambda}{6} \right)^2 t^2 + \frac{1}{u} \left(\frac{\lambda}{6} \right)^2 u^2 \Leftrightarrow \left(\frac{\lambda}{6} \right)^2 (s + t + u) = 0 \quad (2.16)$$

And thus, the scattering amplitude for \mathcal{A}_3 is $\mathcal{A} = 0$.

3 Quadratic and d'Alembertian Derivatives

Lastly, we would like to calculate the scattering amplitude for the following Lagrangian:

$$\mathcal{L}_3 = \frac{1}{\Lambda^3} (\partial\phi)^2 \square\phi, \quad (3.1)$$

where $\square = \partial_\mu \partial^\mu$. Following our previous examples, the first derivative w.r.t ϕ gives us

$$\frac{\delta \mathcal{L}_3}{\delta \phi} = \frac{1}{\Lambda^3} (2(-ip_i^\mu) \partial_\mu \phi \square\phi + (\partial\phi)^2 (-ip_i^\nu) (-ip_{i,\nu})), \quad (3.2)$$

where we've employed Leibniz's rule. Next we take the second derivative which takes on the form

$$\frac{\delta^2 \mathcal{L}_3}{\delta \phi^2} = \frac{1}{\Lambda^3} (2(-ip_i^\mu) (-ip_{j,\mu}) \square\phi + 2(-ip_i^\mu) \partial_\mu \phi (-ip_j^\nu) (-ip_{j,\nu}) + 2(-ip_j^\mu) \partial_\mu \phi (-ip_i^\nu) (-ip_{i,\nu})). \quad (3.3)$$

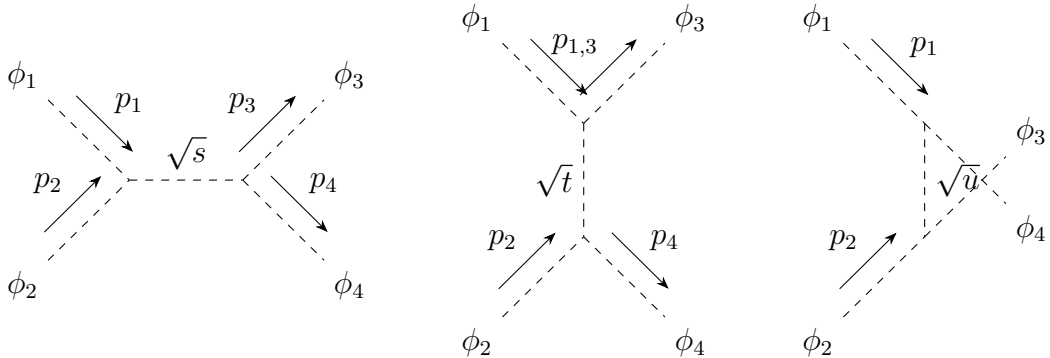
And finally, taking the third derivative gives us

$$\begin{aligned} \frac{\delta^3 \mathcal{L}_3}{\delta \phi^3} = \frac{1}{\Lambda^3} & (2(-ip_i^\mu) (-ip_{j,\mu}) (-ip_k^\nu) (-ip_{k,\nu}) + 2(-ip_i^\mu) (-ip_{k,\mu}) (-ip_j^\nu) (-ip_{j,\nu}) \\ & + 2(-ip_j^\mu) (-ip_{k,\mu}) (-ip_i^\nu) (-ip_{i,\nu})). \end{aligned} \quad (3.4)$$

Collecting all the terms and simplifying a bit gets us

$$\frac{\delta^3 \mathcal{L}_3}{\delta \phi^3} = \frac{2}{\Lambda^3} (p_i \cdot p_j \cdot p_k^2 + p_j \cdot p_k \cdot p_i^2 + p_k \cdot p_i \cdot p_j^2). \quad (3.5)$$

We shall rewrite the diagrams from the previous section in order for us to get a clearer picture as to what we wish to calculate.



Going through the same procedure as we did in the quadratic derivative case, we find that the vertex rule for each diagram is

$$\alpha_s = -\frac{s^2}{\Lambda^3}, \quad \alpha_t = \frac{t^2}{\Lambda^3}, \quad \alpha_u = \frac{(s+t)^2}{\Lambda^3}. \quad (3.6)$$

And remembering that the scattering amplitude is nothing but the sum of the square of the vertex divided by the channel, we get

$$\mathcal{A} = \frac{\alpha_s^2}{s} + \frac{\alpha_t^2}{t} + \frac{\alpha_u^2}{u} \Leftrightarrow -\frac{3st}{\Lambda^6}(s+t) \quad (3.7)$$

And thus, we have the scattering amplitude for this Lagrangian.