Compton Scattering For Spin-s Particles

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We are interested in calculating the amplitude and the cross section for Compton scattering for particles of differing spins. Compton scattering being 2→2 interactions between of boson and some other particle. If we wish to tackle gravitational scattering off of particles of varying spins, it'll be important for us to review the language by which we will conduct our analysis. We take the usual canonical/second quantization route for analyzing scattering (as this is the method that is most familiar to the author) and cover the cases for scalar/vector and spinor/vector scattering as warm ups for the real thing: scalar/tensor, vector/tensor, and spinor/tensor.

This document is organized as follows: section 2 will be dedicated to writing down the respective Lagrangians, equations of motion, and quantizations for a complex scalar, vector, and spinor field. In section 3, we shall focus on putting in interactions with a particular focus to those relevant for Compton scattering. We will then dedicate the last section to gravitational couplings.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_P = \frac{1}{\sqrt{8\pi G}} \approx 2.43 \times 10^{18}\,\text{GeV}$. The d'Alembert and Laplace operators are defined to be $\Box \equiv \partial_\mu \partial^\mu = -\partial_t^2 + \nabla^2$ and $\nabla^2 = \partial_i \partial^i$ respectively. We use boldface letters $\mathbf{r}$ to indicate 3-vectors and x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll.

1 Scalar, Vector, and Spinor Field Canonical Analysis

1.1 Complex Scalar Field

Consider the Lagrangian density

$$\mathcal{L} = -\partial_\mu \Phi^\dagger \partial^\mu \Phi - M^2 \Phi^\dagger \Phi, \tag{1.1}$$

where Φ is a complex scalar field. Note: no factor of 1/2 since a complex number can be written as a combination of two real scalar fields i.e. $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. We can plug this decomposition into the Lagrangian to get

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi_1 \partial^\mu \phi_1 - \frac{1}{2}M^2\phi_1^2 - \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2}M^2\phi_2^2, \tag{1.2}$$

which is just a theory with two scalar fields. Thus, one complex scalar field is equivalent to two real scalar fields. We can write down the action for the complex scalar field as

$$S = \int \mathrm{d}^4x\, \mathcal{L}[\Phi, \Phi^\dagger, \partial\Phi, \partial\Phi^\dagger]. \tag{1.3}$$

We take the field and its complex conjugate to be independent of one another. As such, under the independent variations $\Phi \to \Phi + \delta\Phi$ and $\Phi^\dagger \to \Phi^\dagger + \delta\Phi^\dagger$ with both variations going to zero at the boundary. The action varies as $S \to S + \delta S$ and we require $\delta S = 0$ (the variational principle). As in the real case, we integrate by parts the terms $\partial_\mu \Phi \partial^\mu\, \delta\Phi^\dagger$, $\partial_\mu \Phi^\dagger \partial^\mu\, \delta\Phi$ and drop the surface terms

$$\delta S = \int \mathrm{d}^4x\, [(-\Box\Phi + M^2\Phi)\, \delta\Phi^\dagger + (-\Box\Phi^\dagger + M^2\Phi^\dagger)\, \delta\Phi], \tag{1.4}$$

implies that the equations of motion for both fields are

$$\frac{\partial \mathcal{L}}{\partial \Phi^\dagger} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\dagger)} = -\Box\Phi + M^2\Phi = 0, \quad \frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = -\Box\Phi^\dagger + M^2\Phi^\dagger = 0. \tag{1.5}$$

The solutions are called the "Normal Modes": $\Phi(\mathbf{r},t) = \alpha_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{r}-E_{\mathbf{p}}t)}$. Plugging this into the differential equations gives us

$$\left(-E_{\mathbf{p}}^2 + \mathbf{p}^2 + M^2\right)\Phi(\mathbf{r},t) = 0 \Rightarrow E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + M^2}, \tag{1.6}$$

which looks a lot like a relativistic dispersion relation. Next we normalize the solutions in a cube of side L with periodic boundary conditions (we'll eventually take $L \to \infty$) such that

$$\int \mathrm{d}^3 r \left| \frac{e^{i(\mathbf{p}\cdot\mathbf{r}-E_{\mathbf{p}}t)}}{\sqrt{\mathcal{V}}} \right|^2 = 1. \tag{1.7}$$

This is the discrete momentum representation used in Statistical Mechanics with

$$\sum_{\mathbf{p}} = \sum_{n_x,n_y,n_z\in\mathbb{Z}} \to \mathcal{V}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}. \tag{1.8}$$

The quantized solutions are

$$\hat{\Phi}(\mathbf{r},t) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}} \frac{1}{\sqrt{2E_{\mathbf{p}}}}(\hat{a}_{\mathbf{p}} e^{ip\cdot x} + \hat{b}_{\mathbf{p}}^\dagger e^{-ip\cdot x}), \quad \hat{\Phi}^\dagger(\mathbf{r},t) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}} \frac{1}{\sqrt{2E_{\mathbf{p}}}}(\hat{a}_{\mathbf{p}}^\dagger e^{-ip\cdot x} + \hat{b}_{\mathbf{p}} e^{ip\cdot x}), \tag{1.9}$$

where $p\cdot x = \eta^{\mu\nu}x_\mu p_\nu = \mathbf{p}\cdot\mathbf{r} - E_{\mathbf{p}}t$, and $\hat{a}, \hat{b}$ are independent operators with

$$\left[\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}^\dagger\right] = \delta_{\mathbf{pq}} = \left[\hat{b}_{\mathbf{p}}, \hat{b}_{\mathbf{q}}^\dagger\right], \quad \left[\hat{a}, \hat{b}\right] = \left[\hat{a}, \hat{a}\right] = \left[\hat{b}, \hat{b}\right] = \ldots = 0. \tag{1.10}$$

We can compute the Hamiltonian for this system. First the conjugate momentum is

$$\Pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \dot{\Phi}^\dagger \Rightarrow \Pi^\dagger(x) = \dot{\Phi}. \tag{1.11}$$

And so the quantized Hamiltonian is thus

$$\widehat{\mathcal{H}} = \hat{\Pi}\dot{\hat{\phi}} + \hat{\Pi}^\dagger\dot{\hat{\Phi}}^\dagger - \mathcal{L} = \hat{\Pi}^\dagger\hat{\Pi} + \widehat{\nabla\Phi}^\dagger \cdot \widehat{\nabla\Phi} + M^2\hat{\Phi}^\dagger\hat{\Phi}, \tag{1.12}$$

with

$$\hat{\Pi}^\dagger(x) = \dot{\hat{\Phi}} = \frac{-i}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}\sqrt{\frac{E_{\mathbf{p}}}{2}}(\hat{a}_{\mathbf{p}}e^{ip\cdot x} - \hat{b}_{\mathbf{p}}^\dagger e^{-ip\cdot x}). \tag{1.13}$$

Terms $\hat{a}\hat{a}, \hat{b}\hat{b}, \hat{a}^\dagger\hat{a}^\dagger, \hat{b}^\dagger\hat{b}^\dagger$ cancel out and leaves us with

$$\hat{H} = \int \mathrm{d}^3r\, \hat{\mathcal{H}} = \sum_{\mathbf{p}} \omega_{\mathbf{p}} \left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}} + \frac{1}{2} + \frac{1}{2} \right), \tag{1.14}$$

where the two 1/2's come from the two oscillators. We can also define a charge and momentum operator

$$\hat{Q} = \sum_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} - \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}, \quad \hat{\mathbf{P}} = \sum_{\mathbf{p}} \mathbf{p}(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}}), \tag{1.15}$$

and we can see that $\left[\hat{\mathbf{P}}, \hat{H}\right]$. Note: the value of the charge for the $\hat{a}$ oscillators is positive and for the $\hat{b}$ oscillators is negative. We interpret this as the existence of particles and anti-particles.

Hilbert Space: Fock Representation Because these are creation and annihilation operators, we have the usual relations $\hat{a}_{\mathbf{p}} |0\rangle = \hat{b}_{\mathbf{p}} |0\rangle = 0,\ \forall \mathbf{p} \Rightarrow |0\rangle$ is the vacuum state. We can also represent an arbitrary state as

$$\frac{(\hat{a}_{\mathbf{p}})^{n_{\mathbf{p}}}}{\sqrt{n_{\mathbf{p}}!}} |0\rangle = |n_{\mathbf{p}}\rangle \Rightarrow \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} |n_{\mathbf{p}}\rangle = n_{\mathbf{p}} |n_{\mathbf{p}}\rangle, \tag{1.16}$$

where $|n_{\mathbf{p}}\rangle$ is a state with $n_{\mathbf{p}}$ particles (for a fixed $\mathbf{p}$). We also have $\hat{Q} |n_{\mathbf{p}}\rangle = n_{\mathbf{p}} |n_{\mathbf{p}}\rangle$. We also have

$$\frac{(\hat{b}_{\mathbf{p}})^{\bar{n}_{\mathbf{p}}}}{\sqrt{\bar{n}_{\mathbf{p}}!}} |0\rangle = |\bar{n}_{\mathbf{p}}\rangle, \tag{1.17}$$

which is a state of $\bar{n}_{\mathbf{p}}$ antiparticles in it and

$$\hat{Q} |\bar{n}_{\mathbf{p}}\rangle = -\bar{n}_{\mathbf{p}} |\bar{n}_{\mathbf{p}}\rangle, \tag{1.18}$$

which is the opposite sign of the particles. So we can conclude that particles have the same energy as anti-particles $E_{\mathbf{p}}$, but opposite charge. Using commutation relations yields

$$\left[\hat{Q}, \hat{a}_{\mathbf{p}}\right] = -\hat{a}_{\mathbf{p}}, \quad \left[\hat{Q}, \hat{a}^\dagger_{\mathbf{p}}\right] = \hat{a}^\dagger_{\mathbf{p}}, \quad \left[\hat{Q}, \hat{b}_{\mathbf{p}}\right] = \hat{b}_{\mathbf{p}}, \quad \left[\hat{Q}, \hat{b}^\dagger_{\mathbf{p}}\right] = -\hat{b}^\dagger_{\mathbf{p}}. \tag{1.19}$$

Now consider $\theta =$ constant and

$$e^{-i\theta\hat{Q}}\hat{\Phi}(x)e^{i\theta\hat{Q}} = \hat{\Phi} + i\theta\left[\hat{\Phi}, \hat{Q}\right] + \frac{(-i\theta)^2}{2!}\left[\hat{Q}, \left[\hat{Q}, \hat{\Phi}\right]\right] + \ldots, \tag{1.20}$$

and $\left[\hat{Q}, \hat{\Phi}\right] =$ -$\hat{\Phi}(x)$ which gives us

$$e^{-i\theta\hat{Q}}\hat{\Phi}(x)e^{i\theta\hat{Q}} = \hat{\Phi} + i\theta\hat{\Phi} + \frac{(i\theta)^2}{2!}\hat{\Phi} + \ldots = e^{i\theta}\hat{\Phi}(x). \tag{1.21}$$

The phase, or global gauge transformation forms a group. In this case it is the unitary group $U(1)$: under two consecutive transformations by θ_1 and θ_2 and define the (group) operator $\hat{U}(\theta)$ so that $\hat{U}(\theta)\hat{\Phi}\hat{U}^{\dagger}(\theta) = e^{i\theta}\hat{\Phi}$ which we have identified as $\hat{U}(\theta) = e^{-i\theta\hat{Q}}$. We can see that this is an Abelian group[1] by $\hat{U}(\theta_1)\hat{U}(\theta_2) = e^{-i\theta_1\hat{Q}}e^{-i\theta_2\hat{Q}} = e^{-i(\theta_1+\theta_2)\hat{Q}} = \hat{U}(\theta_2)\hat{U}(\theta_1)$. The group of phase transformations that keep $\mathcal{L}$ is a $U(1)-$Abelian group and the generator is the conserved charge. The Noether conserved charges are the generators of the symmetry group of transformations.

Next we can construct the Green's Function for the Klein-Gordon differential operator. Suppose we have an external source $J(t, \mathbf{r})$ coupled to a complex scalar field

$$\mathcal{L} = -\partial_\mu\Phi\partial^\mu\Phi - M^2\Phi^2 - J\Phi \Rightarrow -\Box\Phi + M^2\Phi = J. \tag{1.22}$$

We can write the solution using a Green's function $G(x-y)$ that satisfies the differential equation

$$-\Box G(x-y) + M^2 G(x-y) = -\delta^{(4)}(x-y) = -\delta(t-\tau)\delta^{(3)}(\mathbf{r}-\mathbf{s}). \tag{1.23}$$

Then, the solution to the equations of motion are

$$\Phi(x) = \Phi_0(x) - \int \mathrm{d}^4y\, G(x-y)J(y), \tag{1.24}$$

with $-\Box\Phi_0 + M^2\Phi_0 = 0$. Find $G(x-y)$ via a spacetime Fourier Transform

$$G(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\tilde{G}(p), \tag{1.25}$$

[1]An Abelian group is a group where the group elements commute under the group operation.

with $p^\mu = (p^0, \mathbf{p})$ and $\mathrm{d}^4p = \mathrm{d}p^0\, \mathrm{d}^3p$ and

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)} = \delta^{(4)}(x-y). \tag{1.26}$$

Inserting these Fourier Transforms into the differential equation yields

$$(p^2 + M^2)\tilde{G}(p) = -1, \tag{1.27}$$

where $p^2 = (p^0)^2 - \mathbf{p}^2$. Then

$$G(x-y) = \int_{\mathbb{R}} \frac{\mathrm{d}p^0}{2\pi} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \frac{e^{-ip^0(t-\tau)} e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{(p^0)^2 - \mathbf{p}^2 + M^2}. \tag{1.28}$$

The problem with this expression is that the $\int \mathrm{d}p^0$ features singularities at $p^0 = \pm\sqrt{\mathbf{p}^2 + M^2} = E_{\mathbf{p}}$ along the real axis which is the range of integration. We need a prescription to handle these singularities i.e. different ways of "going around" the poles at $p^0 = \pm E_{\mathbf{p}}$. In the p^0- Complex plane

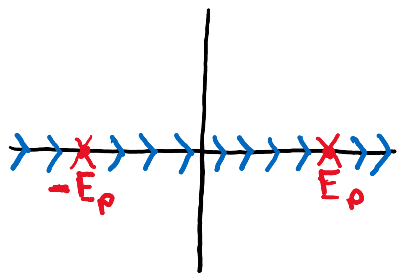


Figure 1: The singularities $\pm E_{\mathbf{p}}$ in the Complex p^0-plane.

Consider the following contour deformations

These contour deformations are equivalent to the "$i\epsilon-$prescriptions" with $\epsilon \to 0^+$. The Green's function becomes

$$\tilde{G}^{(1)}(p^0, \mathbf{p}) = \frac{1}{(p^0 - i\epsilon)^2 - E_{\mathbf{p}}^2}, \quad \tilde{G}^{(2)}(p^0, \mathbf{p}) = \frac{1}{(p^0 + i\epsilon)^2 - E_{\mathbf{p}}^2} \tag{1.29}$$

so the poles are at $p^0 = \pm E_{\mathbf{p}} + i\epsilon$ and $p^0 = \pm E_{\mathbf{p}} - i\epsilon$ respectively. Next we have

$$\tilde{G}^{(3)}(p) = \frac{1}{(p^0)^2 - E_{\mathbf{p}}^2 - i\epsilon} = \frac{1}{(p^0)^2 - (E_{\mathbf{p}}^2 + i\epsilon)}. \tag{1.30}$$

Here the poles are at

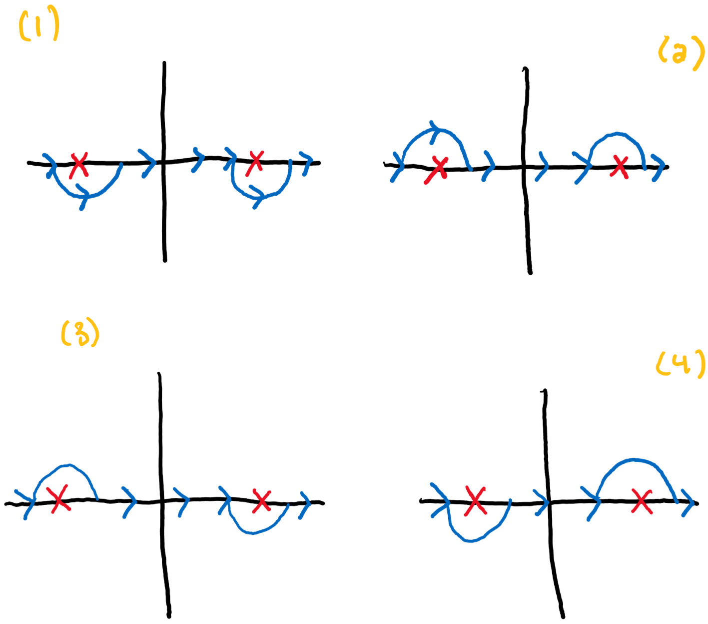


Figure 2: Contour deformation around the singularities in the complex p^0-plane. Each contour results in a different Green's function.

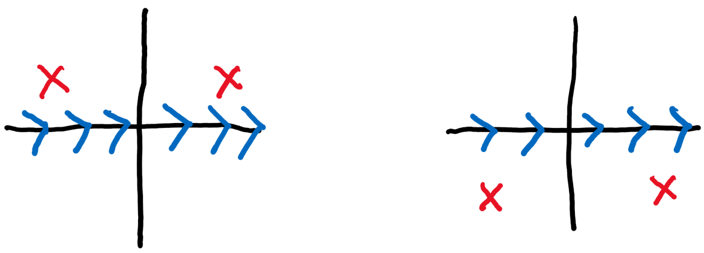


Figure 3: Contour deformation making use of the "$i\epsilon-$prescriptions". By moving the poles off of the real axis, we can compute the contour integral.

$$p^0 = \pm\sqrt{E_{\mathbf{p}}^2 + i\epsilon} = \pm E_{\mathbf{p}}\sqrt{1 + \frac{i\epsilon}{E_{\mathbf{p}}^2}} \approx \pm E_{\mathbf{p}}\left(1 + \frac{i\epsilon}{2E_{\mathbf{p}}^2}\right) = \pm E_{\mathbf{p}}\left(1 + \frac{\eta}{E_{\mathbf{p}}}\right), \tag{1.31}$$

where $\eta = \frac{\epsilon}{2E_{\mathbf{p}}} \to 0^+$. So there are two poles $p^0 = E_{\mathbf{p}} + i\eta$, and $p^0 = -E_{\mathbf{p}} - i\eta$ and $\eta \to 0^+$.

The last propagator is at

$$\tilde{G}^{(4)}(p) = \frac{1}{(p^0)^2 - E_{\mathbf{p}}^2 + i\epsilon} = \frac{1}{(p^0)^2 - (E_{\mathbf{p}}^2 - i\epsilon)}, \tag{1.32}$$

with the poles $p^0 = \pm E_{\mathbf{p}}\left(1 - \frac{i\eta}{E_{\mathbf{p}}}\right) = \pm E_{\mathbf{p}} \mp i\eta$.

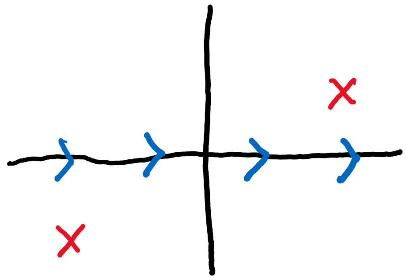


Figure 4: When applying the iϵ prescription to the poles in the Fourier Transform of the Green's function.

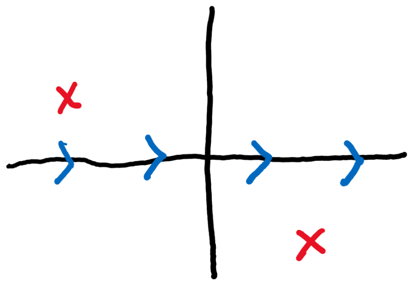


Figure 5: The placement of the poles for the 4-th Green's function Fourier Transform.

1.1.1 Carrying Out The Integrals

All four cases have the form

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p^0}{2\pi} e^{-ip^0(t-\tau)} \tilde{G}(p^0, \mathbf{p}). \tag{1.33}$$

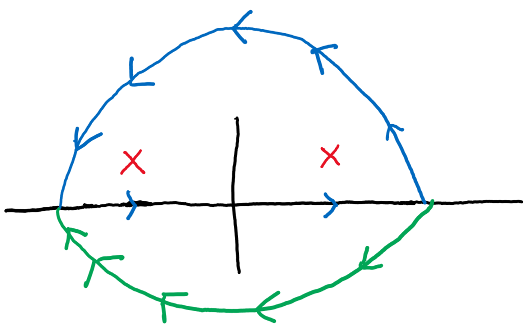


Figure 6: Contour integral around the singularities $\pm E_{\mathbf{p}} + i\epsilon$. The upper half part of the Complex p^0-plane (i.e. $\operatorname{Im} p^0 > 0$) is the region when $t - \tau < 0$. Closing the contour in the bottom half plane ($\operatorname{Im}\{p^0\} < 0$) is done when $t > \tau$.

For $t - \tau < 0$, we close the contour in the upper half plane (UHP) because $\operatorname{Im} p^0 > 0$ which implies there are two poles (and we let $\epsilon \to 0$). Implement the Residue Theorem and get

$$\frac{2\pi i}{2\pi}\left[\frac{e^{-iE_{\mathbf{p}}(t-\tau)}}{2E_{\mathbf{p}}}-\frac{e^{iE_{\mathbf{p}}(t-\tau)}}{2E_{\mathbf{p}}}\right], \tag{1.34}$$

where the first time is from the pole at $E_{\mathbf{p}}+i\epsilon$ and the second term is from the pole at $E_{\mathbf{p}}-i\epsilon$. For $t-\tau>0$, we must close in the lower half plane (LHP) since $\operatorname{Im} p^0<0 \Rightarrow$ there are no poles in the LHP. Thus we can write

$$\int_{\mathbb{R}}\frac{\mathrm{d}p^0}{2\pi}\tilde{G}(p^0,\mathbf{p})e^{ip^0(t-\tau)}=\frac{\Theta(t-\tau)\sin(E_{\mathbf{p}}(t-\tau))}{E_{\mathbf{p}}}. \tag{1.35}$$

In the integrals over $\mathbf{p}$, we can write

$$-i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left[\frac{e^{iE_{\mathbf{p}}(t-\tau)}e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{2E_{\mathbf{p}}}-\frac{e^{-iE_{\mathbf{p}}(t-\tau)}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{2E_{\mathbf{p}}}\right]. \tag{1.36}$$

Taking the second term and relabeling $\mathbf{p}\to-\mathbf{p}$, we can conclude

$$G^{(1)}(x-y)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\Theta(t-\tau)\sin(p\cdot(x-y))}{E_{\mathbf{p}}}. \tag{1.37}$$

We refer to this as the *Advanced Green's Function.*

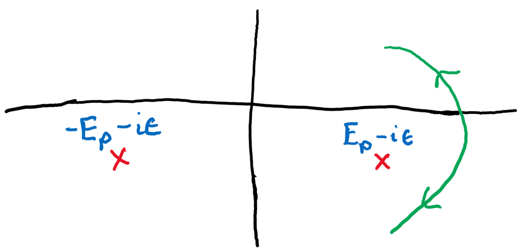


Figure 7: Here, closing in the upper half plane ensures $\tau>t$ and closing in the lower half plane results in $t>\tau$.

Here, both of the poles are in the LHP. This implies that the contour for $\tau>t$ is zero and for $t>\tau$ (where we orient the contour clockwise) we get $-2\pi i\times$ sum of the residue which yields

$$\frac{(-2\pi i)}{2\pi}\left[\frac{e^{iE_{\mathbf{p}}(t-\tau)}e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{2E_{\mathbf{p}}}-\frac{e^{-iE_{\mathbf{p}}(t-\tau)}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{2E_{\mathbf{p}}}\right]\Theta(t-\tau)=-\frac{\sin(E_{\mathbf{p}}(t-\tau))\Theta(t-\tau)}{E_{\mathbf{p}}}, \tag{1.38}$$

where the first term from the $E_{\mathbf{p}}-i\epsilon$ pole and the second term comes from the $-E_{\mathbf{p}}-i\epsilon$ pole. Doing the same treatment as before yields the *Retarded Green's Function*

$$G^{(2)}(x-y) = -\int \frac{d^3p}{(2\pi)^3} \frac{\sin(p\cdot(x-y))}{E_{\mathbf{p}}}\Theta(t-\tau). \tag{1.39}$$

The third Green's function is:

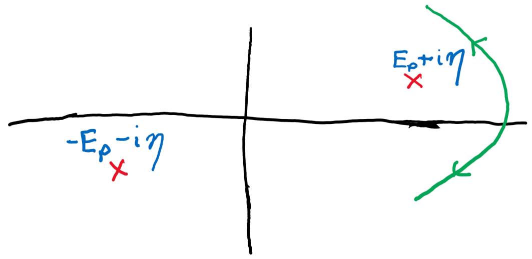


Figure 8: The poles that we integrate around now become $\pm E_{\mathbf{p}} \pm i\eta$ and we the closing in the upper and lower half plane retain their same meanings as previously indicated.

For $t > \tau$, we close the contour in the LHP around the pole at $-E_{\mathbf{p}} - i\eta$ ($\eta \to 0^+$) we orient the contour clockwise and get the pole $\frac{-2\pi i}{2\pi}\frac{e^{iE_{\mathbf{p}}(t-\tau)}}{(-2E_{\mathbf{p}})}$. For $\tau > t$, close in the UHP and orient the contour clockwise around the pole $E_{\mathbf{p}} + i\eta \Rightarrow \frac{2\pi i}{2\pi}\frac{e^{-iE_{\mathbf{p}}(t-\tau)}}{2E_{\mathbf{p}}}$ and the sum of the residues become

$$i\left[\frac{e^{-iE_{\mathbf{p}}(t-\tau)}}{2E_{\mathbf{p}}}\Theta(\tau-t) + \frac{e^{iE_{\mathbf{p}}(t-\tau)}}{2E_{\mathbf{p}}}\Theta(t-\tau)\right], \tag{1.40}$$

and the Green's function becomes

$$G^{(3)}(x-y) = i\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{2E_{\mathbf{p}}}\left[e^{iE_{\mathbf{p}}(t-\tau)}\Theta(t-\tau) + e^{-iE_{\mathbf{p}}(t-\tau)}\Theta(\tau-t)\right]. \tag{1.41}$$

The last Green's function is:

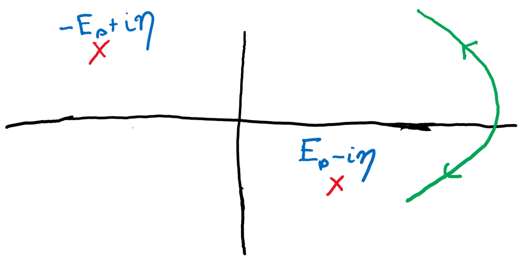


Figure 9: Finally the contour here is for the poles at $\pm E_{\mathbf{p}} \mp i\eta$ with closing in either the upper or lower half of the plane retaining the same meaning as the other contours.

For $t > \tau$, close in LHP (clockwise) around the pole $E_{\mathbf{p}} - i\eta \Rightarrow \frac{-2\pi i}{2\pi}\frac{e^{-iE_{\mathbf{p}}(t-\tau)}}{2E_{\mathbf{p}}}$ and for $\tau > t$ close in the UHP (counterclockwise) pole at $-E_{\mathbf{p}} + i\eta \Rightarrow \frac{2\pi i}{2\pi}\frac{e^{-E_{\mathbf{p}}(t-\tau)}}{(-2E_{\mathbf{p}})}$. And the residue theorem yields

$$-\frac{i}{2E_{\mathbf{p}}}\big[e^{-iE_{\mathbf{p}}(t-\tau)}\Theta(t-\tau)+e^{iE_{\mathbf{p}}(t-\tau)}\Theta(\tau-t)\big], \tag{1.42}$$

and finally we have the *Time-Ordered* or *Feynman Green's Function/Feynman Propagator*

$$G^{(4)}(x-y) = -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{2E_{\mathbf{p}}}\big[e^{-iE_{\mathbf{p}}(t-\tau)}\Theta(t-\tau)+e^{iE_{\mathbf{p}}(t-\tau)}\Theta(\tau-t)\big]. \tag{1.43}$$

Green's functions differ by a solution of the homogeneous equation. This can be shown from, the relation $\Theta(t-\tau)+\Theta(\tau-t)=1$ and the fact that $e^{\pm iE_{\mathbf{p}}(t-\tau)}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}$ are particular solutions of the homogeneous equation $(-\Box+M^2)G=0$.

The Feynman or time ordered Green's function plays a fundamental role in perturbation theory.

1.1.2 Feynman Propagator

In the second term in $G^{(4)}$, relabel $\mathbf{p}\to-\mathbf{p}$ in the d^3p

$$G^{(4)}(x-y)\equiv G_F(x-y) = -i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\big(e^{-i\omega_{\mathbf{p}}(t-\tau)}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}\Theta(t-\tau)+e^{i\omega_{\mathbf{p}}(t-\tau)}e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}\Theta(\tau-t)\big) \tag{1.44}$$

$$= \int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip^0(t-\tau)}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{p^2-M^2+i\epsilon}. \tag{1.45}$$

Recall

$$\langle 0|\,\hat\phi(\mathbf{r},\tau)\hat\phi(\mathbf{s},\tau)\,|0\rangle = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{-iE_{\mathbf{p}}(t-\tau)}e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{2\omega_{\mathbf{p}}},\qquad \langle 0|\,\hat\phi(\mathbf{s},\tau)\hat\phi(\mathbf{r},\tau)\,|0\rangle = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{iE_{\mathbf{p}}(t-\tau)}e^{-i\mathbf{p}\cdot(\mathbf{r}-\mathbf{s})}}{2E_{\mathbf{p}}}, \tag{1.46}$$

which implies that the Feynman Green's function can be written as

$$iG_F(x-y) = \Theta(t-\tau)\,\langle 0|\,\hat\phi(\mathbf{r},\tau)\hat\phi(\mathbf{s},\tau)\,|0\rangle+\Theta(\tau-t)\,\langle 0|\,\hat\phi(\mathbf{s},\tau)\hat\phi(\mathbf{r},t)\,|0\rangle \equiv \langle 0|\,\mathcal{T}(\hat\phi(\mathbf{r},t)\hat\phi(\mathbf{s},\tau))\,|0\rangle\,, \tag{1.47}$$

which is the time-ordered correlation function (or the time ordered product).

1.2 (Massless) Vector Field

E&M can be deduced from the theory of a massless spin-1 vector field. Maxwell's Equations are then obtained from a variational principle for fields similar to the case studied before. The Lagrangian is

$$\mathcal{L}[A, \partial A] = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^{\mu} A_{\mu}, \tag{1.48}$$

where $F_{\mu\nu} = g_{\mu\lambda} g_{\nu\rho} F^{\lambda\rho} = -(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})$. $\mathcal{L}$ is invariant under Lorentz Transformations: $F^{\mu\nu} \to \Lambda\Lambda F^{\alpha\beta}$, $F_{\mu\nu} \to \tilde{\Lambda}\tilde{\Lambda} F_{\alpha\beta} \Rightarrow F^{\mu\nu} F_{\mu\nu}$ is invariant and $J^{\mu} \to \Lambda^{\mu}{}_{\alpha} J^{\alpha}$, $A_{\mu} \to \tilde{\Lambda}^{\alpha}{}_{\mu} A_{\alpha} \Rightarrow J^{\mu} A_{\mu}$ is invariant. Thus

$$S = \int \mathrm{d}^4 x\, \mathcal{L}, \tag{1.49}$$

is invariant under Lorentz Transforms with $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$.

Next we introduce the variational principle $A_{\mu} \to A_{\mu} + \delta A_{\mu}$ with $\delta A_{\mu} \xrightarrow{|t|,|\mathbf{r}| \to \infty} 0$. Under this variation $F \to F + \delta F \Rightarrow$ two linear terms $\delta(F^{\mu\nu})\, F_{\mu\nu} + F^{\mu\nu}\, \delta(F_{\mu\nu})$, but these are the same because for two tensors $F^{\mu\nu} \Delta_{\mu\nu} = F_{\mu\nu} \Delta^{\mu\nu} \Rightarrow \delta\mathcal{L} = -\frac{1}{2} F^{\mu\nu}\, \delta(F_{\mu\nu}) = -\frac{1}{2} F^{\mu\nu}(\partial_{\nu}\, \delta A_{\mu} - \partial_{\mu}\, \delta A_{\nu})$. We can relabel $\mu \leftrightarrow \nu$ in the second term since all indices are summed to get $F^{\nu\mu} \partial_{\nu}\, \delta A_{\mu}$. Thus the Lagrangian and hence the action is

$$\delta\mathcal{L} = -F^{\mu\nu} \partial_{\nu}\, \delta A_{\mu} \Rightarrow \delta S = \int \mathrm{d}^4 x\, [-F^{\mu\nu} \partial_{\nu}\, \delta A_{\mu} + J^{\mu}\, \delta A_{\mu}] = \delta S = \int \mathrm{d}^4 x\, [\partial_{\nu} F^{\mu\nu} + J^{\mu}]\, \delta A_{\mu}\,. \tag{1.50}$$

And we are thus left with the equations of motion $\partial_{\nu} F^{\mu\nu} = -J^{\mu}$. So ME are both covariant and form invariant with Lagrangian $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^{\mu} A_{\mu}$.

Gauge Symmetry The description of the physical fields $\mathbf{E}, \mathbf{B}$ in terms of A^{μ} is redundant:

$$A^0(x) \to A^0(x) - \partial_t \Lambda(x), \quad \mathbf{A}(x) \to \mathbf{A}(x) + \nabla\Lambda(x), \tag{1.51}$$

and both equations can be bundled up into a single equation written as $A^{\mu}(x) \to A^{\mu}(x) + \partial^{\mu}\Lambda$ where Λ is an arbitrary differentiable function. The field strength tensor is varies under this transformation as

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu \to \partial^\nu (A^\mu + \partial^\mu \Lambda) - \partial^\mu (A^\nu + \partial^\nu \Lambda) = F^{\mu\nu} + \partial^\mu \partial^\nu \Lambda - \partial^\nu \partial^\mu \Lambda = F^{\mu\nu}, \tag{1.52}$$

and so the field strength and by extension the electric and magnetic fields are invariant under this transformation.

Under these local gauge transformations, we can use this invariance to reduce the number of fields (A^μ) to the physically relevant ones describing physical degrees of freedom (dof). Let us begin with the source-free ($J^\mu = 0$) theory. The reduction to the physical dof implies a constraint on the gauge fields A^μ. There are several possible constraints that we can place.

Coulomb Gauge: $\nabla \cdot \mathbf{A} = 0$.

For $J^\mu = 0 \Rightarrow \nabla \cdot \mathbf{E} = 0 \Rightarrow -\nabla^2 A^0 - \partial_t (\nabla \cdot \mathbf{A}) \Rightarrow \nabla^2 A^0 = 0$. Imposing regularity at $|\mathbf{r}| \to \infty$, the only regular solution is A^0 =constant which corresponds to a constant potential that can be set to zero which we shall do so ourselves $A^0 = 0$ and $\nabla \cdot \mathbf{A} = 0 \Rightarrow \partial_\mu A^\mu = 0$. Then the equations of motion in the Coulomb gauge become

$$\partial_\nu F^{\mu\nu} = 0 = \partial^\mu \partial_\nu A^\nu - \partial_\nu \partial^\nu A^\mu \Rightarrow \Box A^\mu = 0. \tag{1.53}$$

Since we have $A^0 = 0$, this equation implies

$$\Box \mathbf{A}(\mathbf{r}, t) = 0. \tag{1.54}$$

The solutions to these equations are plane waves:

$$(\partial_t^2 - \nabla^2)\mathbf{A} = 0 \Rightarrow \mathbf{A}(\mathbf{r}, t) = \epsilon(p) e^{-i(p^0 t - \mathbf{p}\cdot\mathbf{r})} \Rightarrow p_0^2 - \mathbf{p}^2 = 0 \Rightarrow p^0 \equiv \omega_\mathbf{p} = p. \tag{1.55}$$

Define $p^\mu = (p^0, \mathbf{p}) \equiv (\omega_\mathbf{p}, \mathbf{p}) \Rightarrow p^\mu x_\mu = \omega_\mathbf{p} t - \mathbf{p}\cdot\mathbf{r}$ and $p^\mu p_\mu = (p^0)^2 - \mathbf{p}^2 = 0 \Rightarrow p^\mu$ is a null 4-vector.

The condition that $-\nabla^2 A^0 = 0$ leaves the possibility of A^0 being solely a function of time: $A^0(t)$. However, this can be "gauged" away by a transformation with a function $\Lambda(t)$ i.e.

$$A'^0(t) = A^0(t) + \partial_t \Lambda(t) = 0 \Rightarrow \Lambda(t) = -\int_0^t A^0(t')\, \mathrm{d}t' \,. \tag{1.56}$$

For this plane wave solution, the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0 \Rightarrow \mathbf{p} \cdot \epsilon = 0$ implies that ϵ is perpendicular to $\mathbf{p} \Rightarrow \epsilon$ is transverse. Define $\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\mathbf{p}}$ as the right-handed triad which acts as a basis in 3D Euclidean space.

Completeness of the Basis

$$\hat{\epsilon}_1^i \hat{\epsilon}_1^j + \hat{\epsilon}_2^i \hat{\epsilon}_2^j + \hat{\epsilon}_3^i \hat{\epsilon}_3^j = \delta^{ij} = [\mathbb{1}_3]^{ij}, \quad \hat{\epsilon}_1 \times \hat{\epsilon}_2 = \hat{\mathbf{p}}, \quad \hat{\epsilon}_2 \times \hat{\mathbf{p}} = \hat{\epsilon}_1, \quad \hat{\mathbf{p}} \times \hat{\epsilon}_1 = \hat{\epsilon}_2. \tag{1.57}$$

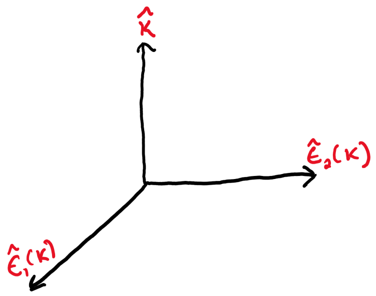


Figure 10: Basis describing the propagation of photons in terms of its wave vector $\hat{\mathbf{k}}$ and polarization vectors that are transverse to its direction of propagation.

All of these properties combine together to yield

$$\sum_{\lambda=1}^{2} \hat{\epsilon}_\lambda^i \hat{\epsilon}_\lambda^j = \delta^{ij} - \hat{\mathbf{p}}^i \hat{\mathbf{p}}^j, \tag{1.58}$$

which is the transverse projection operator and

$$\hat{\epsilon}_\lambda \cdot \hat{\epsilon}_\rho = \delta_{\lambda\rho}. \tag{1.59}$$

More On The Coulomb Gauge: $A^0 = 0,\ \nabla \cdot \mathbf{A} = 0$

Only 2 physical dof corresponding to two independent transverse polarizations $\mathbf{E} = -\partial_t \mathbf{A},\ \mathbf{B} = \nabla \times \mathbf{A}$. This gauge condition (constraint) exhibits the correct physical dof. However, it is frame dependent since under a Lorentz Transform: $A^\mu(x) \to A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(x)$ may not be in the Coulomb gauge. In the new frame, we can do a gauge transformation back to the Coulomb Gauge so Coulomb Gauge + gauge transformation = Coulomb Gauge in any frame. E&M is covariant under Lorentz Transformation and gauge invariant for the physical dof $\mathbf{E}, \mathbf{B}$.

The Coulomb Gauge with $J^\mu \neq 0$ with $\nabla \cdot \mathbf{A} = 0$ and $\mathbf{E} = -\nabla A^0 - \partial_t \mathbf{A} \Rightarrow \nabla \cdot \mathbf{E} = -\nabla^2 A^0 = \rho$. The solution is

$$A^0(\mathbf{r}, t) = \int \mathrm{d}^3 s \, \frac{\rho(\mathbf{s}, t)}{|\mathbf{r} - \mathbf{s}|}, \tag{1.60}$$

this implies $A^0(\mathbf{r}, t)$ is not a dynamical variable but is completely determined by the charge density $\rho(\mathbf{r}, t)$. This leads to the conclusion that there are only two dofs.

Another important gauge condition is the Landau/Lorenz gauge: $\partial_\mu A^\mu = 0$. The virtue of this gauge condition is that it is manifestly Lorentz invariant. In this gauge, the ME become

$$\partial_\nu F^{\mu\nu} = -\partial_\nu \partial^\nu A^\mu + \partial^\mu(\partial_\nu A^\nu) = J^\mu \Rightarrow -\Box A^\mu = J^\mu. \tag{1.61}$$

Compatible with the LL gauge is

$$\Box(\partial_\mu A^\mu) = 0 = \partial_\mu J^\mu. \tag{1.62}$$

The LL gauge leaves 3 dof ($4A^\mu - 1$ constraint). However, LL doesn't completely fix the gauge since we can still perform gauge transformations $A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$ so that $\Box\Lambda = 0$. Such gauge transformation keeps A'^μ in LL but represents a redundancy. We can use this residual gauge transformation to get rid of one dof leaving two physical dof as in the Coulomb Gauge. To see this, consider $J^\mu = 0 \Rightarrow \Box A^\mu = 0,\ \partial_\mu A^\mu = 0$. The plane wave solutions are $A^\mu(x) = \epsilon^\mu(p)e^{-ip\cdot x}$ with $p \cdot p = 0$. The LL enforces $p_\mu \epsilon^\mu = 0$ with $p_\mu = (p_0, -\mathbf{p})$ can write (since $p^2 = 0$)

$$\epsilon^\mu(p) = a(p)p^\mu + b(p)\epsilon^\mu_{(1)} + c(p)\epsilon^\mu_{(2)}, \tag{1.63}$$

where $\epsilon^\mu_{(1)} = (0, \hat{\epsilon}_{(1)}), \epsilon^\mu_{(2)} = (0, \hat{\epsilon}_{(2)})$ and $\hat{\epsilon}_{1,2}$ are the transverse unit vectors that define the Coulomb Gauge under a Gauge Transformation $A^\mu \to A'^\mu = A^\mu + \partial^\mu \Lambda$. In Fourier Space:

$$\tilde{A}^\mu(p) \to \tilde{A}'^\mu(p) = \tilde{A}^\mu(p) + p^\mu \tilde{\Lambda}(p). \tag{1.64}$$

We can fix $\tilde{\Lambda}(p)$ to be $-a(p)$ to cancel the p^μ component of ϵ^μ. As a result, the residual gauge transformation with Λ being a harmonic function ($\Box\Lambda = 0$) to get rid of one dof leaving only the 2 physical dof in the Coulomb gauge. The advantage of LL is that it is Lorentz Invariant and yields a simple equation of motion $-\Box A^\mu = J^\mu$ and $\partial_\mu A^\mu = 0$, but "hides" the correct 2 physical dof. The Coulomb Gauge exhibits explicitly the two physical dof: transverse polarizations. The action is invariant under Gauge Transformations: under $A^\mu(x) \to A^\mu + \partial^\mu \Lambda$ with $\Lambda \xrightarrow{|\mathbf{r}|, t \to \infty} 0$. Thus from the action

$$S = \int \mathrm{d}^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu \right], \tag{1.65}$$

under the gauge transformation

$$S \to S' = \int \mathrm{d}^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu + J^\mu \partial_\mu \Lambda \right]. \tag{1.66}$$

And the last term goes to zero under integration by parts as well as charge conservation. Moving back to the Coulomb Gauge where $A^0 = 0 \Rightarrow \dot{A}^i = -E^i$, the Hamiltonian Density then becomes

$$\mathcal{H} = \Pi^0 \dot{A}_0 - \Pi^i \dot{A}^i - \mathcal{L} = \frac{1}{2}\left(\mathbf{E}^2 + \mathbf{B}^2\right), \tag{1.67}$$

where we used the fact that $\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)$. The Hamiltonian then becomes

$$H = \frac{1}{2} \int \mathrm{d}^3r \left(\mathbf{E}^2 + \mathbf{B}^2\right). \tag{1.68}$$

In the Coulomb Gauge (when $J^\mu = 0$) $\Box \mathbf{A} = 0,\ A^0 = 0$. The particular solution (called the normal modes) is $A^i(x) = \epsilon^i(p) e^{-ip\cdot x}$ which yields

$$\hat{A}^i(\mathbf{r}, t) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p}} \sum_{\lambda=1}^{2} \frac{\epsilon^i_\lambda(p)}{\sqrt{2\omega_{\mathbf{p}}}} \left[\hat{a}_{\mathbf{p},\lambda} e^{ip\cdot x} + \hat{a}^\dagger_{\mathbf{p},\lambda} e^{-ip\cdot x} \right], \tag{1.69}$$

where $p \cdot x = -\omega_{\mathbf{p}} t + \mathbf{p} \cdot \mathbf{r},\ \omega_{\mathbf{p}} = |\mathbf{p}|$ and $\mathbf{E} = -\partial_t \mathbf{A},\ \mathbf{B} = \nabla \times \mathbf{A}$, with $\hat{\epsilon}_{1,2}$, defined previously. The Hamiltonian then becomes

$$\hat{H} = \sum_{\mathbf{p}} \sum_{\lambda=1}^{2} \left(\hat{a}^\dagger_{\mathbf{p},\lambda} \hat{a}_{\mathbf{p},\lambda} + \frac{1}{2} \right), \tag{1.70}$$

with the commutation relations

$$\left[\hat{a}_{\mathbf{p},\lambda}, \hat{a}^\dagger_{\mathbf{q},\rho}\right] = \delta_{\mathbf{pq}} \delta_{\lambda\rho}, \quad [\hat{a}, \hat{a}] = \left[\hat{a}^\dagger, \hat{a}^\dagger\right] = 0, \quad \hat{a}_{\mathbf{p},\lambda} |0\rangle = 0, \tag{1.71}$$

in which $\hat{a}_{\mathbf{p},\lambda}$ annihilates a photon of momentum $\mathbf{p}$ and polarization λ and $\hat{a}^\dagger_{\mathbf{p},\lambda}$ creates a photon of $\mathbf{p}$ and polarization λ. We can then represent any Fock state with $n_{\mathbf{p},\lambda}$ photons of momentum $\mathbf{p}$ and polarization λ by

$$|n_{\mathbf{p},\lambda}\rangle = \frac{(\hat{a}^\dagger_{\mathbf{p},\lambda})^{n_{\mathbf{p},\lambda}}}{\sqrt{n_{\mathbf{p},\lambda}!}} |0\rangle. \tag{1.72}$$

1.2.1 The Photon Propagator

Recall the Euler-Lagrange Equations for E&M

$$-\partial_\nu F^{\mu\nu} = J^\mu \Rightarrow \partial_\nu(-\partial^\nu A^\mu + \partial^\mu A^\nu) = J^\mu \Leftrightarrow \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = J^\mu. \tag{1.73}$$

The photon Green's Function/propagator will be used in Feynman calculus to obtain gauge invariant observables, such as cross-sections or transition rates. Therefore we can calculate in any gauge since the result is independent of such choices. Lets start with the LL gauge: $\partial_\nu A^\nu = 0$:

$$-\Box A^\mu = J^\mu. \tag{1.74}$$

Now we introduce the Green's function $G^{\mu\nu}(x-y)$ that satisfies

$$-\Box_x G^{\mu\nu}(x-y) = g^{\mu\nu}\delta^{(4)}(x-y). \tag{1.75}$$

Thus we can write the solution to the equations of motion by

$$A^\mu(\mathbf{r},t) = \int \mathrm{d}^4y\, G^{\mu\nu}(x-y)J_\nu(y). \tag{1.76}$$

Just as for the scalar field, introduce the 4D Fourier transform

$$G^{\mu\nu}(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\tilde{G}^{\mu\nu}(p), \qquad \delta^{(4)}(x-y) = \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{ip\cdot(x-y)}. \tag{1.77}$$

Fourier Transforming the differential equation for the Green's function gives

$$p^2\tilde{G}^{\mu\nu} = g^{\mu\nu} \Rightarrow \tilde{G}^{\mu\nu} = \frac{g^{\mu\nu}}{p^2}. \tag{1.78}$$

Again we need to specify the pole prescription:

(I) $p^0 \to p^0 - i\epsilon \Rightarrow$ Advanced

(II) $p^0 \to p^0 + i\epsilon \Rightarrow$ Retarded

(III) $(p^0)^2 - \mathbf{p}^2 \to (p^0)^2 - \mathbf{p}^2 + i\epsilon = p^2 + i\epsilon \Rightarrow$ Feynman.

This is the important G_F to calculate cross sections etc. in Feynman calculus! Thus the Green's function is then

$$\tilde{G}^{\mu\nu}(p) = \frac{g^{\mu\nu}}{p^2 + i\epsilon}. \tag{1.79}$$

The main results of the description with sources in the Coulomb Gauge are the following. ME in the Coulomb Gauge becomes

$$\nabla \cdot \mathbf{E} = -\nabla^2 A^0 = J^0. \tag{1.80}$$

A spatial Fourier Transform yields

$$\tilde{A}^0(p) = \frac{\tilde{J}^0(p)}{\mathbf{p}^2} \Rightarrow \tilde{G}^{00} = \frac{1}{\mathbf{p}^2}. \tag{1.81}$$

Next we have

$$-\Box \mathbf{A}(\mathbf{r}, t) = \mathbf{J}(\mathbf{r}, t) \Rightarrow \tilde{\mathbf{A}}(p) = \frac{1}{p^2 + i\epsilon} \tilde{\mathbf{J}}(p), \tag{1.82}$$

or it can also be written as

$$\tilde{A}^i = \frac{\delta^{ij}}{p^2 + i\epsilon} \tilde{J}_j(p), \tag{1.83}$$

where we've lowered the index j. Note that this only holds for $\nabla \cdot \mathbf{A} = 0$. The other components of the Green's function in this gauge are

$$\tilde{G}^{0i} = \tilde{G}^{i0} = 0, \quad \tilde{G}^{ij} = \frac{\delta^{ij}}{p^2 + i\epsilon}, \tag{1.84}$$

where i, j only correspond to components obeying $\mathbf{p} \cdot \mathbf{A}^T = \mathbf{p} \cdot \mathbf{J}^T = 0$. Similarly to the scalar field case, we can relate the time-ordered product to the Feynman propagator of the photon

$$\left\langle 0 \middle| \mathcal{T}(\hat{A}^\mu(\mathbf{r}, t) \hat{A}^\nu(\mathbf{s}, \tau)) \middle| 0 \right\rangle \equiv i G_F^{\mu\nu}(x - y) = i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{g^{\mu\nu}}{p^2 + i\varepsilon}. \tag{1.85}$$

1.3 Dirac Spinor

Now we focus on massive Dirac spinors. We start off from the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi, \tag{1.86}$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^0$. We work in the representation where,

$$\gamma^0 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix}, \quad \gamma_i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}. \tag{1.87}$$

We vary $\psi^\dagger, \psi$ independently $\psi \to \psi + \delta\psi\,, \psi^\dagger \to \psi^\dagger + \delta\psi^\dagger$ with $\delta\psi\,, \delta\psi^\dagger \xrightarrow{|r|,t\to\infty} 0$ and $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$. Let us consider

$$\mathcal{L} = \psi^\dagger\big(i\partial_t\psi + i\alpha^i\partial_i\psi - \beta m\psi\big), \tag{1.88}$$

where $\beta = \gamma^0$ and $\alpha^i = \gamma^0\gamma^i$ as the Lagrangian and

$$S = \int \mathrm{d}^4x\,\mathcal{L}, \tag{1.89}$$

is the action. Under the variations $\psi \to \psi + \delta\psi\,,\ \psi^\dagger \to \psi^\dagger + \delta\psi^\dagger$ and assuming $\delta S = 0$ to linear order, the variation $\psi^\dagger \to \psi^\dagger + \delta\psi^\dagger$ gives us

$$\delta S = \int \mathrm{d}^4x\,\delta\psi^\dagger\,\big[i\partial_t\psi + i\alpha^i\partial_i\psi - \beta m\psi\big] = 0, \tag{1.90}$$

and under the variation $\psi \to \psi + \delta\psi$ we have

$$\delta S = \int \mathrm{d}^4x\,\psi^\dagger\big[i\partial_t\,\delta\psi + i\alpha^i\partial_i\,\delta\psi\big] = \int \mathrm{d}^4x\,\big[-i\partial_t\psi^\dagger - i\partial^j\psi^\dagger\alpha^j - \psi^\dagger\beta m\big]\,\delta\psi\,. \tag{1.91}$$

This yields the equation of motion

$$(i\partial\!\!\!/ - m)\psi = 0. \tag{1.92}$$

The solutions of the Dirac Equation are 4-component Dirac spinors

$$\psi(\mathbf{r}, t) = e^{-iEt}e^{i\mathbf{p}\cdot\mathbf{r}}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} \equiv \begin{bmatrix} u_A \\ u_B \end{bmatrix}, \tag{1.93}$$

and the Dirac Equation

$$\big(i\hbar\partial_t + i\alpha^i\partial_i - \beta m\big)\psi = 0. \tag{1.94}$$

where u_A and u_B given by

$$u_A = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \quad u_B = \begin{bmatrix} \psi_3 \\ \psi_4 \end{bmatrix}, \tag{1.95}$$

are the two-component spinors just like in Non-Relativistic Quantum Mechanics of spin-1/2 electrons. In the standard Dirac Representation

$$\begin{bmatrix} (E-m)\mathbb{1}_2 & -p^i\sigma_i \\ -p^i\sigma_i & (E+m)\mathbb{1}_2 \end{bmatrix} \begin{bmatrix} u_A \\ u_B \end{bmatrix} = 0. \tag{1.96}$$

This matrix equation yields two additional equations

$$(E-m)u_A = (p^i\sigma_i)u_B, \quad (E+m)u_B = (p^i\sigma_i)u_A. \tag{1.97}$$

Using the fact that $(p^i\sigma_i)(p^j\sigma_j) = \mathbf{p}^2$ by multiplying the leftmost equation by $(\vec{\sigma}\cdot\vec{\mathbf{p}})$ (as well as using the rightmost equation), we get the dispersion relation

$$E^2 - \mathbf{p}^2 = m^2 \Rightarrow E = \pm\sqrt{\mathbf{p}^2 + m^2}. \tag{1.98}$$

We therefore have two solutions for the energy. With this knowledge we can solve for u_B in terms of u_A to get

$$u_B = \frac{p^i\sigma_i}{E+m}u_A \xrightarrow{\mathbf{p}\to 0} 0. \tag{1.99}$$

Thus, we have two independent two-component spinors

$$u_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{1.100}$$

for the positive energy solution taking the form

$$u_{1,2} = \mathcal{N}\begin{bmatrix} u_A \\ \frac{p^i\sigma_i}{E+mc^2}u_A \end{bmatrix}, \tag{1.101}$$

where u_A are the two component spinors defined previously. The negative energy solutions $E = -\sqrt{\mathbf{p}^2 + m^2}$ of the Dirac Equation can be found in a similar way

$$u_A = -\frac{p^i \sigma_i}{|E| + m} u_B \xrightarrow{\mathbf{p} \to 0} 0, \tag{1.102}$$

where $E = -|E|$. We get the same two linearly independent spinors with the 4 component spinors being

$$u_{3,4} = \mathcal{N} \begin{bmatrix} -\frac{p^i \sigma_i}{|E|+m} u_B \\ u_B \end{bmatrix}, \tag{1.103}$$

where

$$u_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{1.104}$$

For $\mathbf{p} = 0$ (the rest frame of the electron) we get

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad u_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{1.105}$$

where u_1 is the positive energy spin up in the rest frame of the electron (spin along z-axis which would therefore make the two-component spinor to be an eigenvector of σ_z), u_2 is the spin down positive energy spinor along the z-axis, u_3 is the negative energy spin up spinor and u_4 is the negative energy spin down spinor.

Dirac realized that the negative energy solutions correspond to anti-particles

$$p^i \sigma_i = \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix}. \tag{1.106}$$

It is convenient to define the 4 solutions as

$$u_1(\mathbf{p}) = \mathcal{N}_{\mathbf{p}} \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{p^i\sigma_i}{|E|+m} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \mathcal{N}_{\mathbf{p}} \begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{|E|+m} \\ \frac{(p_x+ip_y)}{|E|+m} \end{bmatrix}, \tag{1.107}$$

$$u_2(\mathbf{p}) = \mathcal{N}_{\mathbf{p}} \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \frac{p^i\sigma_i}{|E|+m} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \mathcal{N}_{\mathbf{p}} \begin{bmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{|E|+m} \\ \frac{-p_z}{|E|+m} \end{bmatrix}. \tag{1.108}$$

For $E = -|E|$, we write

$$v_1(\mathbf{p}) = u_4(-\mathbf{p}) = \mathcal{N}_{\mathbf{p}} \begin{bmatrix} \frac{p^i\sigma_i}{|E|+m} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \mathcal{N}_{\mathbf{p}} \begin{bmatrix} \frac{p_x-ip_y}{|E|+m} \\ \frac{-p_z}{|E|+m} \\ 0 \\ 1 \end{bmatrix}, \tag{1.109}$$

$$v_2(\mathbf{p}) = u_3(-\mathbf{p}) = -\mathcal{N}_{\mathbf{p}} \begin{bmatrix} \frac{p^i\sigma_i}{|E|+m} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \mathcal{N}_{\mathbf{p}} \begin{bmatrix} \frac{p_z}{|E|+m} \\ \frac{p_x+ip_y}{|E|+m} \\ 1 \\ 0 \end{bmatrix}, \tag{1.110}$$

where we normalize our solutions by

$$u_A^\dagger u_B = v_A^\dagger v_B = 2|E|\delta_{AB}, \tag{1.111}$$

where $u^\dagger = (u_1^*, u_2^*, u_3^*, u_4^*)$. Further more, it is straightforward to prove that

$$u_A^\dagger(\mathbf{p}) v_B(-\mathbf{p}) = 0, \tag{1.112}$$

for $A, B = 1, 2$. The normalization condition yields

$$\mathcal{N}_{\mathbf{p}}^2 \left(1 + \frac{\mathbf{p}^2}{(|E|+m)^2}\right) = 2|E| \Rightarrow \mathcal{N}_{\mathbf{p}} = \sqrt{|E|+m}. \tag{1.113}$$

The 4 "canonical" solutions $u_1(\mathbf{p}), u_2(\mathbf{p}), v_1(\mathbf{p}), v_2(\mathbf{p})$ form a complete set of solutions to the Dirac Equation.

Current Conservation Writing the equations

$$i\partial_t\psi = (-i\alpha^i\partial_i + \beta m)\psi, \quad i\partial_t\psi^\dagger = i\partial_j\psi^\dagger\alpha^j + \psi^\dagger\beta m, \tag{1.114}$$

then multiplying the first equation by $\psi^\dagger$ and the second equation by ψ and subtracting them we get

$$i\partial_t(\psi^\dagger\psi) = i\partial_i(\psi^\dagger\alpha^i\psi). \tag{1.115}$$

Define the Dirac charge density and current as

$$\rho \equiv \psi^\dagger\psi, \quad J^i \equiv \psi^\dagger\alpha^i\psi \Rightarrow \partial_t\rho + \nabla\cdot\mathbf{J} = 0 \Leftrightarrow \partial_\mu J^\mu, \tag{1.116}$$

where $J^\mu = (\rho, \mathbf{J})$. Now we can define $\overline{\psi} \equiv \psi^\dagger\gamma^0$ which frees us up to write $\psi^\dagger\alpha^i\psi = \overline{\psi}\gamma^0\alpha^i\psi$ which yields $\rho = \overline{\psi}\gamma^0\psi,\ J^i = \overline{\psi}\gamma^i\psi$ thus

$$J^\mu = \overline{\psi}\gamma^\mu\psi. \tag{1.117}$$

The Lagrangian is invariant under global gauge transformation $\psi(x) \to e^{i\theta}\psi(x),\ \psi^\dagger(x) \to \psi^\dagger(x)e^{-i\theta}$ where $\theta \in \mathbb{R}$. Consider the infinitesimal variation $\delta\psi = i\theta\psi,\ \delta\psi^\dagger = -i\psi^\dagger\theta$. From Noether's Theorem we can construct the current

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^\dagger)}\,\delta\psi^\dagger = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta\psi = \overline{\psi}\gamma^\mu\psi, \tag{1.118}$$

which is conserved. A result previously obtained from the Dirac Equation. We can also see that under a Lorentz Transformation $\overline{\psi} \to \overline{\psi}S^{-1},\ \psi \to S\psi$ which implies

$$J^\mu \to \overline{\psi}S^{-1}\gamma^\mu S\psi = \overline{\psi}\Lambda^\mu{}_\nu\gamma^\nu\psi = \Lambda^\mu{}_\nu J^\nu. \tag{1.119}$$

And so we can conclude that J^μ transforms as a contravariant 4-vector.

1.4 Quantization

Since $\psi(x)$ is a complex field, the most general solution of the Dirac Equation is a complex linear superposition of particular solutions

$$\hat{\psi}(\mathbf{r},t) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{A=1}^{2}\sum_{\mathbf{p}}\frac{1}{\sqrt{E_{\mathbf{p}}}}(\hat{b}_{\mathbf{p}A}u_A(\mathbf{p})e^{ip\cdot x} + \hat{d}^{\dagger}_{\mathbf{p}A}v_A(\mathbf{p})e^{-ip\cdot x}). \tag{1.120}$$

The factor of $(2E_{\mathbf{p}})^{-1/2}$ is there because we normalized the spinors u, v such that $u^{\dagger}u = v^{\dagger}v = 2|E_{\mathbf{p}}|$, $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, $\hat{b}_{\mathbf{p}A}, \hat{d}^{\dagger}_{\mathbf{p}A}$ are the generalized Fourier coefficients. Using the equations

$$\delta_{\mathbf{pq}} = \frac{1}{\mathcal{V}}\int \mathrm{d}^3r\,\exp(i(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}), \tag{1.121}$$

$$(\alpha^i p_i + \beta m)u_A(\mathbf{p}) = E_{\mathbf{p}}u_A(\mathbf{p}), \tag{1.122}$$

$$(-\alpha^i p_i + \beta m)v_A(\mathbf{p}) = -E_{\mathbf{p}}v_A(\mathbf{p}), \tag{1.123}$$

$$u^{\dagger}_A \cdot u_B = v^{\dagger}_A \cdot v_B = 2E_{\mathbf{p}}\delta_{AB}, \quad u^{\dagger}_A(\mathbf{p})v_B(-\mathbf{p}) = v^{\dagger}_B(\mathbf{p})u_A(\mathbf{p}) = 0, \tag{1.124}$$

we get the Hamiltonian to be

$$\hat{H} = \int \mathrm{d}^3r\,\hat{\psi}^{\dagger}(\mathbf{r},t)\big[-i\alpha^i\partial_i + \beta m\big]\hat{\psi}(\mathbf{r},t) = \sum_{\mathbf{p},A}E_{\mathbf{p}}\left[\hat{b}^{\dagger}_{\mathbf{p}A}\hat{b}_{\mathbf{p}A} - \hat{d}_{\mathbf{p}A}\hat{d}^{\dagger}_{\mathbf{p}A}\right], \tag{1.125}$$

the momentum is

$$\hat{\mathbf{P}} = \sum_{\mathbf{p},A}\mathbf{p}[\hat{b}^{\dagger}_{\mathbf{p}A}\hat{b}_{\mathbf{p}A} - \hat{d}_{\mathbf{p}A}\hat{d}^{\dagger}_{\mathbf{p}A}], \tag{1.126}$$

and the charge is

$$\hat{Q} = \int \mathrm{d}^3r\,\hat{J}^0(\mathbf{r},t) = \int \mathrm{d}^3r\,\hat{\psi}^{\dagger}(x)\hat{\psi}(x) = \sum_{\mathbf{p},A}\left[\hat{b}^{\dagger}_{\mathbf{p}A}\hat{b}_{\mathbf{p}A} + \hat{d}_{\mathbf{p}A}\hat{d}^{\dagger}_{\mathbf{p}A}\right]. \tag{1.127}$$

Can we impose canonical commutation relations as for Klein Gordon fields? Doing so leaves us with

$$\left[\hat{d}_{\mathbf{p}A}, \hat{d}^{\dagger}_{\mathbf{q}B}\right] = \delta_{\mathbf{pq}}\delta_{AB} \Rightarrow \hat{d}_{\mathbf{p}A}\hat{d}^{\dagger}_{\mathbf{p}A} = 1 + \hat{d}^{\dagger}_{\mathbf{p}A}\hat{d}_{\mathbf{p}A}, \tag{1.128}$$

which alters the form of the Hamiltonian by

$$\hat{H} = \sum_{\mathbf{p},A} E_{\mathbf{p}} \Big[\hat{b}^\dagger_{\mathbf{p}A}\hat{b}_{\mathbf{p}A} - \hat{d}^\dagger_{\mathbf{p}A}\hat{d}_{\mathbf{p}A} - 1\Big], \tag{1.129}$$

and we get a negative zero point energy. Because of the (-) sign, we can lower the energy by allowing more quanta of negative energy. This implies that the energy is unbounded from below which yields an unstable ground state. To obtain a well defined ground state, we must instead impose anti-commutation relations on the Fourier coefficients

$$\Big\{\hat{b}_{\mathbf{p}A}, \hat{b}^\dagger_{\mathbf{q}B}\Big\} = \Big\{\hat{d}_{\mathbf{p}A}, \hat{d}^\dagger_{\mathbf{q}B}\Big\} = \delta_{\mathbf{pq}}\delta_{AB}, \quad \Big\{\hat{b}, \hat{b}\Big\} = \Big\{\hat{d}, \hat{d}\Big\} = \Big\{\hat{b}^\dagger, \hat{d}^\dagger\Big\} = \Big\{\hat{b}, \hat{d}^\dagger\Big\} = 0. \tag{1.130}$$

Now we have $\hat{d}_{\mathbf{p}A}\hat{d}^\dagger_{\mathbf{p}A} = -\hat{d}^\dagger_{\mathbf{p}A}\hat{d}_{\mathbf{p}A} + 1$ which when plugged into the Hamiltonian becomes

$$\hat{H} = \sum_{\mathbf{p},A} E_{\mathbf{p}} \Big[\hat{b}^\dagger_{\mathbf{p}A}\hat{b}_{\mathbf{p}A} + \hat{d}^\dagger_{\mathbf{p}A}\hat{d}_{\mathbf{p}A}\Big] - 2\sum_{\mathbf{p}} E_{\mathbf{p}}, \tag{1.131}$$

where the 2 results from us doing the A sum. Therefore the ground state $|0\rangle$ leaves

$$\hat{b}_{\mathbf{p}A}|0\rangle = \hat{d}_{\mathbf{p}A}|0\rangle = 0, \tag{1.132}$$

where we've promoted the Fourier coefficients to creation and annihilation operators.

1.4.1 Field Operators

The Dirac field operator is (plane wave normalized)

$$\hat{\psi}(\mathbf{r},t) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}\sum_{A=1}^{2}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\Big[\hat{b}_{\mathbf{p}A}u_A(\mathbf{p})e^{ip\cdot x} + \hat{d}^\dagger_{\mathbf{p}A}v^\dagger_A(\mathbf{p})e^{-ip\cdot x}\Big], \tag{1.133}$$

$$\hat{\psi}^\dagger(\mathbf{r},t) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}}\sum_{A=1}^{2}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\Big[\hat{b}^\dagger_{\mathbf{p}A}u^\dagger_A(\mathbf{p})e^{-ip\cdot x} + \hat{d}_{\mathbf{p}A}v_A(\mathbf{p})e^{ip\cdot x}\Big]. \tag{1.134}$$

ψ and $\psi^\dagger$ are 4-component spinors and obey the equal time anti-commutation relations

$$\Big\{\hat{\psi}_A(\mathbf{r},t), \hat{\psi}^\dagger_B(\mathbf{s},t)\Big\} = \delta_{AB}\delta^{(3)}(\mathbf{r}-\mathbf{s}), \quad \Big[\hat{\psi}_A, \hat{\psi}_B\Big] = \Big\{\hat{\psi}^\dagger_A, \hat{\psi}^\dagger_B\Big\} = 0. \tag{1.135}$$

1.5 Spin Statistics Theorem

For scalar fields (spin 0), photons and massive vector bosons (spin 1), quantization is carried out by imposing commutation relations. These yield a Hamiltonian bounded from below (and are therefore stable) with a well-defined ground state

$$\hat{H} = \sum_{\mathbf{p},\lambda} E_{\mathbf{p}} \left(\hat{a}^{\dagger}_{\mathbf{p}\lambda} \hat{a}_{\mathbf{p}\lambda} + \frac{1}{2} \right). \tag{1.136}$$

For spin-1/2 particles, we must quantize with anti-commutation relations to define a ground state and a Hamiltonian bounded from below with

$$\hat{H} = \sum_{\mathbf{p},A} E_{\mathbf{p}} \left[\hat{b}^{\dagger}_{\mathbf{p}A} \hat{b}_{\mathbf{p}A} + \hat{d}^{\dagger}_{\mathbf{p}A} \hat{d}_{\mathbf{p}A} \right] + \text{zero point energy}, \tag{1.137}$$

Excitations are created by $\hat{b}^{\dagger}, \hat{d}^{\dagger}$ and obey the Pauli Exclusion Principle as a result of the anti-commutator relations.

1.5.1 Spin Statistics

Integer spin fields (particles) with $s = 0, 1, \dots$ must be quantized with canonical commutation relations. These particles are bosons and they are symmetric under an exchange of pairs $\hat{a}^{\dagger}_{\mathbf{p}\lambda} \hat{a}^{\dagger}_{\mathbf{q}\rho} = \hat{a}^{\dagger}_{\mathbf{q}\rho} \hat{a}^{\dagger}_{\mathbf{p}\lambda}$ (from $\left[\hat{a}^{\dagger}, \hat{a}^{\dagger}\right] = 0$). The consequences: the lowest energy $N-$ particle state is a Bose Einstein Condensate (BEC). Half-odd integer spin particles $s = 1/2, 3/2, \dots$ are fermions (Fermi-Dirac Statistics) and must be quantized with anti-commutation relations: particles states are anti-symmetric under pairwise exchange $\hat{b}^{\dagger}_{\mathbf{p}\alpha} \hat{b}^{\dagger}_{\mathbf{q}\beta} = -\hat{b}^{\dagger}_{\mathbf{q}\beta} \hat{b}^{\dagger}_{\mathbf{p}\alpha}$ (from $\left\{b^{\dagger}, b^{\dagger}\right\} = 0$) which results in the Pauli Exclusion Principle.

The spin statistics theorem is proven rigorously in Lorentz invariant relativistic theories with local Lagrangian densities, obeying micro-causality. In fermionic theories, bilinear operators (e.g. charge, density $\psi^{\dagger}\psi$) must commute for space-like intervals! The non-relativistic limit and anti-commuting Fermi fields. The Non Relativistic limit corresponds to the kinetic energy $\ll$ rest energy $\Leftrightarrow p^2 \ll m^4 \Rightarrow E_{\mathbf{p}} \simeq m$.

1.5.2 Time-Ordered Product for Fermions

For fermion (anti-commuting) operators $a(x), b(y)$, the time-ordered product is defined as

$$\mathcal{T}(a(x)b(y)) = a(x)b(y)\Theta(t-\tau) - b(y)a(x)\Theta(\tau - t), \tag{1.138}$$

where the minus sign is there because of the anti-commuting nature of the fermion fields. As a consequence of this we have

$$\mathcal{T}(a(x)b(y)) = -\mathcal{T}(b(y)a(x)). \tag{1.139}$$

Now consider $\left\langle 0\middle|\mathcal{T}\left(\hat{\psi}_B(x)\hat{\overline{\psi}}_C(y)\right)\middle|0\right\rangle$ with $\hat{\psi} \sim \hat{b}u + \hat{d}^\dagger v$, $\hat{\overline{\psi}} = \hat{\psi}^\dagger\gamma^0$, and $\hat{b}\left|0\right\rangle = \hat{d}\left|0\right\rangle = 0$. Since $(i\partial\!\!\!/_x - m)\psi(x) = 0$, we can apply the Dirac operator to the time ordered product to get

$$\begin{aligned}(i\partial\!\!\!/_x - m)\left\langle 0\middle|\mathcal{T}\left(\hat{\psi}_B(x)\hat{\overline{\psi}}_C(y)\right)\middle|0\right\rangle = (i\gamma^0\partial_t + i\gamma^i\partial_{i,x} - m)_{AB}\Big[\left\langle 0\middle|\mathcal{T}\left(\hat{\psi}_B(x)\hat{\overline{\psi}}_C(y)\right)\middle|0\right\rangle\Theta(t-\tau)\\ -\left\langle 0\middle|\mathcal{T}\left(\hat{\overline{\psi}}_C(y)\hat{\psi}_B(x)\right)\middle|0\right\rangle\Theta(\tau-t)\Big].\end{aligned} \tag{1.140}$$

Using the fact that

$$\partial_t\Theta(t-\tau) = \delta(t-\tau), \quad \partial_t\Theta(\tau-t) = -\delta(\tau-t), \tag{1.141}$$

1.140 reduces down to

$$\gamma^0_{AB}\left(\left\langle 0\middle|\hat{\psi}_B(x)\hat{\overline{\psi}}_C(y)\middle|0\right\rangle + \left\langle 0\middle|\hat{\overline{\psi}}_C(y)\hat{\psi}_B(x)\middle|0\right\rangle\right)\delta(t-\tau). \tag{1.142}$$

With $\hat{\overline{\psi}}_C(y) = \hat{\psi}^\dagger_D\gamma^0_{CD} \Rightarrow$ the $i\gamma^0\partial_t$ acting on the Θ-functions yields

$$i\gamma^0_{AB}\gamma^0_{BC}\left\langle 0\middle|\left\{\hat{\psi}_B(x), \hat{\psi}^\dagger_C(y)\right\}\middle|0\right\rangle\delta(t-\tau) = i\gamma^0_{AB}\gamma^0_{BC}\delta^{(3)}(\mathbf{r}-\mathbf{s})\delta(t-\tau) \tag{1.143}$$

$$= i\delta_{AB}\delta^{(3)}(\mathbf{r}-\mathbf{s})\delta(t-\tau). \tag{1.144}$$

Thus when the Dirac operator acts on $\hat{\psi}(x)$ such that $(\partial\!\!\!/ - m)_{AB}\hat{\psi}_B(x) = 0$ we get

$$(i\partial\!\!\!/ - m)_{AB}\left\langle 0\middle|\mathcal{T}\left(\hat{\psi}_B(x)\hat{\bar{\psi}}_C(y)\right)\middle|0\right\rangle = i\delta_{AB}\delta^{(4)}(x-y). \tag{1.145}$$

Comparing to $\tilde{S}_{AB}$ we have

$$\left\langle 0\middle|\mathcal{T}\left(\hat{\psi}_B(x)\hat{\bar{\psi}}_C(y)\right)\middle|0\right\rangle = iS_{BC}(x-y), \tag{1.146}$$

which is the Feynman time-ordered product/Green's function for fermions.

2 Interaction Picture

Now we are ready to deal with interactions. In the presence of interactions in the Schrodinger picture, states evolve as $|\psi(t)\rangle = e^{-i\hat{H}(t-t_i)}|\psi(t_i)\rangle$ with $\hat{H} = \hat{H}_0 + \hat{H}_{int}$. A large part of the time evolution is "free" corresponding to a phase that doesn't affect the transition probability. The interaction picture removes this trivial phase as follows: the total time evolution operator is

$$e^{-i\hat{H}(t-t_i)} = e^{-i\hat{H}_0 t}\hat{U}(t;t_i)e^{i\hat{H}_0 t} \Rightarrow \hat{U}(t;t_i) = e^{i\hat{H}_0 t}e^{i\hat{H}(t-t_i)}e^{-i\hat{H}_0 t}, \tag{2.1}$$

within the interaction picture. This operator is unitary i.e. $\hat{U}^\dagger\hat{U} = \mathbb{1}$. Consider an initial and final state $|i\rangle, |f\rangle$ as Fock states which are also eigenstates of H_0 with eigenvalues E_i, E_f respectively. Then the transition amplitude can be expressed as

$$\mathcal{A}_{i\to f} = \langle f|e^{-i\hat{H}(t_f-t_i)}|i\rangle = \langle f|e^{-i\hat{H}_0 t}\hat{U}(t_f;t_i)e^{i\hat{H}_0 t}|i\rangle = e^{-i(E_f t_f - E_i t_i)}\langle f|\hat{U}(t_f;t_i)|i\rangle \tag{2.2}$$

$$\Rightarrow |\mathcal{A}_{i\to f}|^2 = \mathcal{P}_{i\to f} = |\langle f|\hat{U}(t_f;t_i)|i\rangle|^2. \tag{2.3}$$

The transition probability only depends on the time evolution operator in the interaction picture with $\hat{U}(t_i;t_i) = \mathbb{1}$. So what is $\hat{U}(t,t_i)$? If $[H_0, H_{int}] \neq 0$, then the exponentials can't be combined into a single exponential. Restore $\hbar$ and consider

$$i\hbar\partial_t\hat{U}(t;t_i) = i\hbar\partial_t\Big[e^{\frac{i\hat{H}_0 t}{\hbar}}e^{-\frac{i}{\hbar}\hat{H}(t-t_i)}e^{-\frac{i}{\hbar}\hat{H}_0 t_i}\Big] = i\hbar\left[\frac{i\hat{H}_0}{\hbar}\hat{U}(t;t_i) - \frac{i}{\hbar}e^{\frac{i\hat{H}_0 t}{\hbar}}\hat{H}e^{-\frac{i}{\hbar}\hat{H}_0 t}e^{\frac{i}{\hbar}\hat{H}_0 t}e^{\frac{i}{\hbar}\hat{H}(t-t_i)}e^{-\frac{i}{\hbar}\hat{H}_0 t_i}\right] \tag{2.4}$$

$$= i\hbar\left[\frac{i}{\hbar}\hat{H}_0\hat{U}(t;t_i) - \frac{i}{\hbar}\left(e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{H}e^{-\frac{i}{\hbar}\hat{H}_0 t}\right)\hat{U}(t;t_i)\right] \tag{2.5}$$

$$= e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{H}_{int}e^{-\frac{i}{\hbar}\hat{H}_0 t}\hat{U}(t;t_i) \equiv \hat{H}_I(t)\hat{U}(t;t_i). \tag{2.6}$$

where we regard $\hat{H}_I(t)$ as the interaction Hamiltonian in the interaction picture i.e. the Heisenberg picture of "free fields". Thus the time evolution operator satisfies the following equation

$$\frac{\partial\hat{U}}{\partial t} = -\frac{i}{\hbar}\hat{H}_I(t)\hat{U}(t;t_i), \quad \hat{U}(t_i;t_i) = \mathbb{1}, \quad \hat{H}_I(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{H}_{int}e^{-\frac{i}{\hbar}\hat{H}_0 t}. \tag{2.7}$$

Thus in this interaction Hamiltonian, the fields feature the free field time evolution as in the Heisenberg picture without interactions. In the interaction picture, states evolve in time via the interaction of operators that evolve in time with the free field time evolution. For example in QED we have

$$H_I = e\int\overline{\psi}\not{A}\psi\,\mathrm{d}^3r\,, \quad H_I(t) = e\int\mathrm{d}^3r\,\overline{\psi}(\mathbf{r},t)\not{A}(\mathbf{r},t)\psi(\mathbf{r},t), \tag{2.8}$$

where $\overline{\psi}, \psi, A$ are all free fields. The solution to the evolution equation for the time evolution operator is

$$\hat{U}(t;t_i) = \mathbb{1} - \frac{i}{\hbar}\int_{t_i}^{t}\hat{H}_I(t_1)\hat{U}(t_1;t_i)\,\mathrm{d}t_1\,. \tag{2.9}$$

Since $\hat{H}_I$ depends on a coupling (for QED the coupling is e) assumed small, we can give the solution as a power series expansion in this coupling by iteration: replace

$$\hat{U}(t_1;t_i) = \mathbb{1} - \int_{t_i}^{t_1}\mathrm{d}t_2\,\hat{H}_I(t_2)\hat{U}(t_2;t_i)$$

$$\Rightarrow \hat{U}(t;t_i) = \mathbb{1} - \frac{i}{\hbar}\int_{t_i}^{t}\mathrm{d}t_1\,\hat{H}_I(t_1) + \left(\frac{-i}{\hbar}\right)^2\int_{t_i}^{t_1}\mathrm{d}t_1\int_{t_i}^{t_2}\mathrm{d}t_2\,\hat{H}_I(t_2)\hat{H}_I(t_1)\hat{U}(t_2;t_i), \tag{2.10}$$

and we can express the very last term as

$$\hat{U}(t_2;t_i) = \mathbb{1} - \frac{i}{\hbar}\int_{t_i}^{t_2} \mathrm{d}t_3\, \hat{H}_I(t_3)\hat{U}(t_3;t_i), \tag{2.11}$$

which means we can write

$$\begin{aligned}\hat{U}(t;t_i) = \mathbb{1} - \frac{i}{\hbar}\int_{t_i}^{t} \mathrm{d}t_1\, \hat{H}_I(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_{t_i}^{t}\mathrm{d}t_1 \int_{t_i}^{t_1} \mathrm{d}t_2\, \hat{H}_I(t_1)\hat{H}_I(t_2) \\ + \left(\frac{-i}{\hbar}\right)^3 \int_{t_i}^{t}\mathrm{d}t_1 \int_{t_i}^{t_1}\mathrm{d}t_2 \int_{t_i}^{t_2} \mathrm{d}t_3\, \hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3) + \ldots\end{aligned} \tag{2.12}$$

This is a power series expansion in $\hat{H}_I$ (proportional to coupling) as time-ordered integrals: $\hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3)\cdots$ with $t_1 > t_2 > t_3 > \ldots$.

In a scattering experiment, the initial state is prepared well before the scattering events (far in the past) i.e. $t_i \to -\infty$ and the final state is measured at a detector a few meters away from the collision region, well after the scattering event. For example, at the LHC, particles travel $\sim 10\,\mathrm{km}$ before colliding and detectors are $\sim$ 10 meters away from the collision region and particles move with $v/c \approx 1$. This makes $t_i \sim 10^{-5}\,\mathrm{s}$, $t_f \sim 10^{-8}\,\mathrm{s}$ but the collision "time" $\sim$ of a proton/c $\approx 10^{-28}\,\mathrm{s}$ which means we can safely set $t_i \to -\infty, t_f \to \infty$ and we thus need $S = \hat{U}(\infty, -\infty)$ which is defined to be the S-matrix $S_{fi} \equiv \langle f|\hat{U}(\infty,-\infty)|i\rangle$ which we can write as

$$\begin{aligned}\hat{U}(\infty,-\infty) = \mathbb{1} - \frac{i}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}t_1\, \hat{H}_I(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2\, \hat{H}_I(t_1)\hat{H}_I(t_2) \\ + \left(\frac{-i}{\hbar}\right)^3 \int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{t_i}^{t_1}\mathrm{d}t_2 \int_{t_i}^{t_2} \mathrm{d}t_3\, \hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3) + \ldots.\end{aligned} \tag{2.13}$$

Now we look at the 2nd order term and write it as

$$I = \int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2\, \Theta(t_1 - t_2)\hat{H}_I(t_2). \tag{2.14}$$

Since t_1, t_2 are dummy variables, we can relabel them freely $t_1 \leftrightarrow t_2$ which brings the integral to the new form

$$I = \frac{1}{2!}\int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2\, [\hat{H}_I(t_1)\hat{H}_I(t_2)\Theta(t_1 - t_2) + \hat{H}_I(t_2)\hat{H}_I(t_1)\Theta(t_2 - t_1)] \tag{2.15}$$

$$= \frac{1}{2!}\int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2\, \mathcal{T}(\hat{H}_I(t_1)\hat{H}_I(t_2)). \tag{2.16}$$

This analysis can be done to all orders i.e.

$$\hat{U}(\infty,-\infty) = \mathbb{1} - \frac{i}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}t_1\, \hat{H}_I(t_1) + \frac{1}{2!}\left(\frac{-i}{\hbar}\right)^2 \int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{-\infty}^{\infty}\mathrm{d}t_2\, \mathcal{T}(\hat{H}_I(t_1)\hat{H}_I(t_2)) \\ + \frac{1}{3!}\left(\frac{-i}{\hbar}\right)^3 \int_{-\infty}^{\infty}\mathrm{d}t_1 \int_{-\infty}^{\infty}\mathrm{d}t_2 \int_{-\infty}^{\infty}\mathrm{d}t_3\, \mathcal{T}(\hat{H}_I(t_1)\hat{H}_I(t_2)\hat{H}_I(t_3)) + \ldots \equiv \mathcal{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\hat{H}_I(t)}\right), \tag{2.17}$$

where here we regard $\mathcal{T}$ to be the time-ordering symbol. This inspires us to express the S-matrix element (henceforth the matrix element) as

$$S_{fi} = \left\langle f \middle| \mathcal{T}\left(\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\, \hat{H}_I(t)\right)\right) \middle| i \right\rangle, \tag{2.18}$$

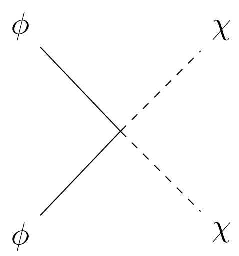
and $S = \hat{U}(\infty,-\infty) = \mathcal{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\hat{H}_I(t)}\right)$ is the full S-matrix. This is the essential ingredient in decay rates and cross sections by featuring a power series expansion in the couplings

$$\hat{U}(\infty,-\infty) = \mathbb{1} + \hat{U}^{(1)}(\infty,-\infty) + \hat{U}^{(2)}(\infty,-\infty) + \ldots, \tag{2.19}$$

where $\hat{U}^{(1)} \sim \mathcal{O}(e)$, $\hat{U}^{(2)} \sim \mathcal{O}(e^2)$ and so on. As a result, the S-matrix is Lorentz and gauge invariant (for relativistic Lorentz invariant Lagrange densities) even when it involves the Hamiltonian.

2.1 2→2 Processes

Consider the theory of two real scalars ϕ, χ with $\mathcal{L}_{int} = -\lambda\phi^2\chi^2$ and $H_i = \lambda\int \mathrm{d}^3r\, \phi^2\chi^2$, the vertex is now



We focus on this case because it is instructive on the general processes and techniques for all interactions of this form. Let us study the following processes: annihilation $\phi\phi \to \chi\chi$ and scattering $\phi\chi \to \phi\chi$. In the interaction picture

$$\hat{\phi}(x) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{p}} \frac{1}{\sqrt{2E^{\phi}_{\mathbf{p}}}}\left[\hat{a}_{\mathbf{p}}e^{ip\cdot x} + \hat{a}^{\dagger}_{\mathbf{p}}e^{-ip\cdot x}\right], \quad \hat{\chi}(x) = \frac{1}{\sqrt{\mathcal{V}}}\sum_{\mathbf{q}} \frac{1}{\sqrt{2E^{\chi}_{\mathbf{q}}}}\left[\hat{b}_{\mathbf{q}}e^{iq\cdot x} + \hat{b}^{\dagger}_{\mathbf{q}}e^{-iq\cdot x}\right], \tag{2.20}$$

the initial state is $\left|\zeta^{\phi}_{p_1}, \zeta^{\phi}_{p_2}\right\rangle$ and the final state is $\left|\zeta^{\chi}_{q_1}, \zeta^{\chi}_{q_2}\right\rangle$. This process features a 1st order contribution $\langle f|\hat{U}^{(1)}(\mathbb{R})|i\rangle = -i\lambda\int_{\mathbb{R}^4} \mathrm{d}^4x\ \langle f|\hat{\phi}^2\hat{\chi}^2|i\rangle$. We need to destroy the initial state $\left|\zeta^{\phi}_{p_1}, \zeta^{\phi}_{p_2}\right\rangle \to |0\rangle$ and create the final state out of the vacuum $\left|\zeta^{\chi}_{q_1}, \zeta^{\chi}_{q_2}\right\rangle$ so it has overlap with $\langle f|$. The relevant term from $\hat{\phi}^2$ is

$$\sum_{\mathbf{k}_1,\mathbf{k}_2} \frac{1}{\sqrt{2\mathcal{V}E^{\phi}_{\mathbf{k}_1}2\mathcal{V}E^{\phi}_{\mathbf{k}_2}}}\left[\hat{a}_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}e^{i(k_1+k_2)\cdot x}\right], \tag{2.21}$$

because we need to annihilate both particles from $|i\rangle$ which means we need two terms $k_1 = p_1, k_2 = p_2$. Now we also need to create $\left|\zeta^{\chi}_{q_1}, \zeta^{\chi}_{q_2}\right\rangle$ so the relevant $\hat{\chi}^2$ term is

$$\sum_{\mathbf{k}_1,\mathbf{k}_2} \frac{1}{\sqrt{2\mathcal{V}E^{\chi}_{\mathbf{k}_1}2\mathcal{V}E^{\chi}_{\mathbf{k}_2}}}\left[\hat{b}_{\mathbf{k}_1}\hat{b}_{\mathbf{k}_2}e^{-i(k_1+k_2)\cdot x}\right], \tag{2.22}$$

and we need the two terms $k_1 = q_1, k_2 = q_2$ and $k_1 = q_2, k_2 = q_1$. Creating $\left|\zeta^{\chi}_{q_1}, \zeta^{\chi}_{q_2}\right\rangle$ which now has $\left\langle f\middle|\zeta^{\chi}_{q_1}, \zeta^{\chi}_{q_2}\right\rangle = 1$ implies the S-matrix element is

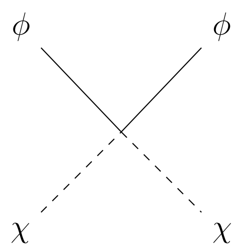
$$S_{fi} = 2\times 2\lambda\int \mathrm{d}^4x\, \frac{e^{i(p_1+p_2-q_1-q_2)\cdot x}}{(2\mathcal{V}E^{\phi}_{\mathbf{p}_1}2\mathcal{V}E^{\phi}_{\mathbf{p}_2}2\mathcal{V}E^{\chi}_{\mathbf{q}_1}2\mathcal{V}E^{\chi}_{\mathbf{q}_2})^{1/2}}, \tag{2.23}$$

where the first factor of 2 is from destroying the initial state and the second factor of 2 is from creating the final state. This leaves the S-matrix element to the form

$$S_{fi} = \left[\prod_{i=1}^{2}(2\mathcal{V}E^{\phi}_{\mathbf{p}_i})^{-1/2}\prod_{i=1}^{2}(2\mathcal{V}E^{\chi}_{\mathbf{q}_i})^{-1/2}\right]\left[(2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_f q_f\right)\mathcal{M}_{fi}\right], \tag{2.24}$$

where the products serve as normalization factors and the delta function enforces energy and momentum conservation. This the main result for 2→2 processes. The scattering amplitude $\mathcal{M}_{fi}$ only depends on the particular interaction vertex and not on the normalizations of E/p conservation. For $\phi\phi \to \chi\chi$, $\mathcal{M}_{fi} = 4\lambda$. For $\lambda\phi^2\chi^2$ scattering process $\phi\chi \to \phi\chi$ we have $|i\rangle = \left|\zeta^{\phi}_{p_1}, \zeta^{\chi}_{q_1}\right\rangle$, $|f\rangle = \left|\zeta^{\phi}_{p_2}, \zeta^{\chi}_{q_2}\right\rangle$

The $\hat{\phi}^2$ terms that are important in destroying the initial state and creating the final state are



$$2\sum_{\mathbf{k}_1,\mathbf{k}_2}\frac{1}{\sqrt{2\mathcal{V}E^{\phi}_{\mathbf{k}_1}2\mathcal{V}E^{\phi}_{\mathbf{k}_2}}}\left[\hat{a}^{\dagger}_{\mathbf{k}_1}\hat{a}_{\mathbf{k}_2}e^{i(k_1-k_2)\cdot x}\right], \tag{2.25}$$

where the factor of 2 is due to there being two terms i.e. $\hat{\phi}^2 \sim (\hat{a}+\hat{a}^{\dagger})(\hat{a}+\hat{a}^{\dagger}) \sim \hat{a}\hat{a}^{\dagger}$. Similarly for $\hat{\chi}^2$, the relevant terms are

$$2\sum_{\mathbf{k}_1,\mathbf{k}_2}\frac{1}{\sqrt{2\mathcal{V}E^{\chi}_{\mathbf{k}_1}2\mathcal{V}E^{\chi}_{\mathbf{k}_2}}}\left[\hat{b}^{\dagger}_{\mathbf{k}_1}\hat{b}_{\mathbf{k}_2}e^{i(k_1-k_2)\cdot x}\right], \tag{2.26}$$

to destroy $\left|\zeta^{\chi}_{q_1}\right\rangle$ and create $\left|\zeta^{\chi}_{q_2}\right\rangle$ to overlap with $\langle f|$ to get

$$S_{fi} = 2\times 2\lambda\int \mathrm{d}^4x\,\frac{e^{i(p_1+q_1-p_2-q_2)\cdot x}}{(2\mathcal{V}E^{\phi}_{\mathbf{p}_1}2\mathcal{V}E^{\phi}_{\mathbf{p}_2}2\mathcal{V}E^{\chi}_{\mathbf{q}_1}2\mathcal{V}E^{\chi}_{\mathbf{q}_2})^{1/2}} \tag{2.27}$$

$$= \left[\prod_{i=1}^{2}(2\mathcal{V}E^{\phi}_{\mathbf{p}_i})^{-1/2}\prod_{i=1}^{2}(2\mathcal{V}E^{\chi}_{\mathbf{q}_i})^{-1/2}\right]\left[(2\pi)^4\delta^{(4)}\left(\sum_i p_i - \sum_f q_f\right)\mathcal{M}_{fi}\right]. \tag{2.28}$$

We can see that

$$|S_{fi}|^2 = \frac{|\mathcal{M}_{fi}|^2}{\mathcal{V}^4}\frac{[(2\pi)^4\delta^{(4)}(p+q_1-p_2-q_2)]^2}{2E^{\phi}_{\mathbf{p}_1}2E^{\phi}_{\mathbf{p}_2}2E^{\chi}_{\mathbf{q}_1}2E^{\chi}_{\mathbf{q}_2}}. \tag{2.29}$$

What is $[(2\pi)^4\delta^{(4)}(p_1+q_1-p_2-q_2)]^2$? Because

$$(2\pi)^4\delta^{(4)}(p_1+q_1-p_2-q_2) = \int \mathrm{d}^4x\, e^{i(p_1+q_1-p_2-q_2)\cdot x} \Rightarrow (2\pi)^4\delta^{(4)}(p_1+q_1-p_2-q_2)\int \mathrm{d}^4x\, e^{i(p_1+q_1-p_2-q_2)\cdot x}. \tag{2.30}$$

Since $\delta^{(4)}(p_1+q_1-p_2-q_2)$ is only nonzero for when $p_1+q_1-p_2-q_2=0$, we set $p_1+q_1-p_2-q_2$ in the integral and write $\int \mathrm{d}^4x \equiv \mathcal{V}T$ which is the total volume of spacetime. This leaves us with the transition probability per unit time

$$\frac{\mathcal{P}_{i\to f}}{T} = \frac{1}{\mathcal{V}^3}\frac{(2\pi)^4\delta^{(4)}(p_1+q_1-p_2-q_2)|\mathcal{M}_{fi}|^2}{2E^{\phi}_{\mathbf{p}_1}2E^{\phi}_{\mathbf{p}_2}2E^{\chi}_{\mathbf{q}_1}2E^{\chi}_{\mathbf{q}_2}}. \tag{2.31}$$

Thus the total probability per unit time is given by the sum over all final states and we also use $\sum_{\mathbf{p}}\sum_{\mathbf{q}} \to \mathcal{V}\int \frac{\mathrm{d}^3 p_2}{(2\pi)^3}\mathcal{V}\int \frac{\mathrm{d}^3 q_2}{(2\pi)^3}$. We must include a symmetry factor $\tilde{S}$ to account for indistinguishability in the final state (in this case $\tilde{S} = 1$ but for 2 identical particles $\tilde{S} = \frac{1}{2!}$). We then have

$$\frac{\mathcal{P}^{tot}_{i\to f}}{T} = \frac{\tilde{S}}{4E^{\phi}_{\mathbf{p}_1}E^{\chi}_{\mathbf{q}_1}\mathcal{V}}\int \frac{\mathrm{d}^3 p_2}{(2\pi)^3 2E^{\phi}_{\mathbf{p}_2}}\int \frac{\mathrm{d}^3 q_2}{(2\pi)^3 2E^{\chi}_{\mathbf{q}_2}}|\mathcal{M}_{fi}|^2(2\pi)^4\delta^{(4)}(p_1+q_1-p_2-q_2). \quad (2.32)$$

2.1.1 Cross Sections

Let A be an incident particle and B be the target with relative velocity v_{rel}. The incident beam features a number density $n_A = \frac{N_A}{\mathcal{V}}$. Consider a cross-sectional area $\Delta a \perp$ to the incident beam

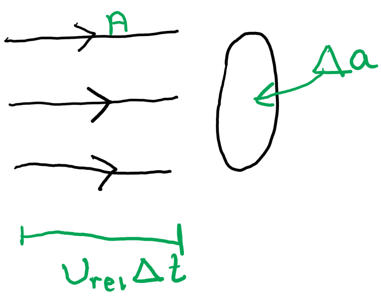


Figure 11: Cylinder of volume $v_{\mathrm{rel}}\Delta t\Delta a$.

In a time ΔT, the number of $A-$particles passing through the area Δa is $\Delta N_A = n_A(v_{\mathrm{rel}}\Delta t)\Delta a$ where $(v_{\mathrm{rel}}\Delta t)\Delta a$ is the volume of particles passing through the cross sectional area Δa in a time Δt.

The flux F, is the number of particles passing per unit area per unit time given by

$$F = \frac{n_A v_{\mathrm{rel}}\Delta t\Delta a}{\Delta a\Delta t} = n_A v_{\mathrm{rel}}. \quad (2.33)$$

If the target presents an effective cross-sectional area σ to the incoming beam then all particles within σ will be scattered off the beam

where σ is approximately the interaction area between incident and target particles. As a result, the number of A-particles scattered off the particles per unit time is equal to the number of incident (A) per unit area per unit time $\times\sigma$ i.e. $F\sigma$. But this is the probability per unit time for the incident particles to transition to a final state

$$F\sigma = \frac{\mathcal{P}_{i\to f}}{T} \Rightarrow \sigma = \frac{\mathcal{P}_{i\to f}}{T} = \frac{1}{F}. \quad (2.34)$$

Now the cross section is defined for a single particle i.e. $n_A = 1/\mathcal{V}$ which leads to

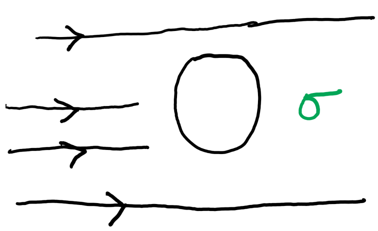


Figure 12: The incoming beam of particles as it comes into contact with the effective cross sectional area σ. Particles that fall inside the cross sectional area scatter off the surface while particles not in the line of sight stream freely due to the lack of contact,

$$\sigma = \left(\frac{\mathcal{V}}{v_{\text{rel}}}\right)\left(\frac{\mathcal{P}_{i\to f}}{T}\right). \tag{2.35}$$

Plugging in the result for the probability per unit time gives us

$$\sigma = \frac{\tilde{S}}{4E_1E_2v_{\text{rel}}}\int\frac{\mathrm{d}^3p_3}{(2\pi)^3 2E_3}\int\frac{\mathrm{d}^3p_4}{(2\pi)^3 2E_4}|\mathcal{M}_{fi}|^2_*(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4), \tag{2.36}$$

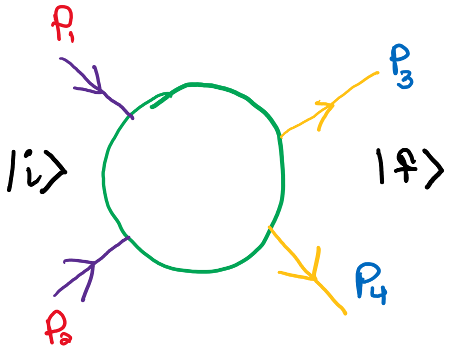


Figure 13: 2→2 where the particles in the initial state $|i\rangle$ start with momenta p_1, p_2 and end in the final state $|f\rangle$ with momenta p_3, p_4.

where we've renamed $q_1 \to p_2$, $p_2 \to -p_3$, $q_2 \to p_4$. Recall $\dfrac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}$ is the Lorentz invariant phase space (LIPS) and for collinear scattering

$$E_1E_2v_{\text{rel}} = \sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}, \tag{2.37}$$

and in the Center of Mass $\mathbf{p}_2 = -\mathbf{p}_1$.

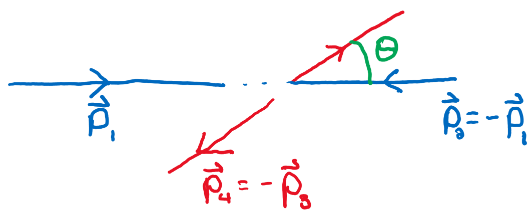


Figure 14: 2→2 scattering in the center of mass frame. The initial momenta of particles 1 and 2 are collinear as well as the final momenta of particles 3 and 4. However, particles 3 and 4 are scattering off an angle with respect to the initial particles.

Thus in the center of mass frame, $E_1E_2v_{\text{rel}} = |\mathbf{p}_1|(E_1+E_2)$ where E_1+E_2 represent the total energy in the center of mass. Scattering in the center of mass frame can be depicted as

Now given that

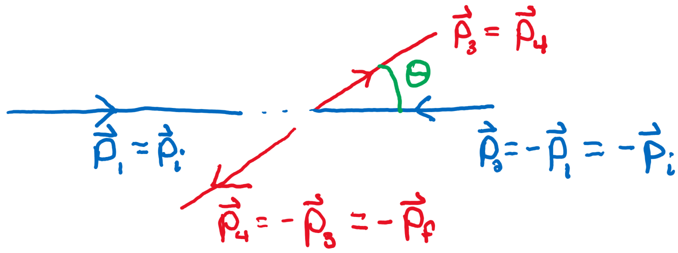


Figure 15: Scattering in the center of mass frame.

$$(p_1 \cdot p_2)^2 - (m_1 m_2)^2 = |\mathbf{p}_i|(E_1 + E_2), \tag{2.38}$$

and recognizing that $\delta^{(4)}(p_1+p_2-p_3-p_4) = \delta(E_1+E_2-E_3-E_4)\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)$ and that $\mathbf{p}_1 + \mathbf{p}_2 = 0$ means that the cross section is

$$\sigma = \frac{\tilde{S}}{(8\pi)^2|\mathbf{p}_i|(E_1+E_2)} \int \frac{\mathrm{d}^3 p_f}{E_3 E_4} \delta(E_1 + E_2 - E_3 - E_4) |\mathcal{M}_{fi}|^2_{\mathbf{p}_3 = -\mathbf{p}_f}. \tag{2.39}$$

Using $\mathrm{d}^3 p_f = p_f^2 \, \mathrm{d}p_f \, \mathrm{d}\Omega$, $\mathrm{d}\Omega = 2\pi \, \mathrm{d}\cos\theta$ and $E_1 + E_2 = E_3 + E_4 = E_{\text{tot}} = \sqrt{s}$ where $s = (p_1 + p_2)^2 = (E_1 + E_2)^2$ in the center of mass. Thus we express the differential cross section

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{CM} = \frac{\tilde{S}}{(8\pi)^2|\mathbf{p}_i|E_{\text{tot}}} \int_0^\infty \frac{p_f^2|\mathcal{M}_{fi}|^2\delta(E_{\text{tot}} - (E_3 + E_4))}{\sqrt{p_f^2 + m_3^2}\sqrt{p_f^2 + m_4^2}}. \tag{2.40}$$

We define $x \equiv \sqrt{p_f^2 + m_3^2} + \sqrt{p_f^2 + m_4^2} \Rightarrow p_f = p_f(x) \Rightarrow$

$$\frac{\mathrm{d}x}{\mathrm{d}p_f} = p_f \left[\frac{1}{\sqrt{p_f^2 + m_3^2}} + \frac{1}{\sqrt{p_f^2 + m_4^2}} \right] = \frac{p_f(x) x}{\sqrt{p_f^2 + m_3^2}\sqrt{p_f^2 + m_4^2}} \Leftrightarrow \frac{p_f \, \mathrm{d}p_f}{\sqrt{p_f^2 + m_3^2}\sqrt{p_f^2 + m_4^2}} = \frac{\mathrm{d}x}{x}. \tag{2.41}$$

We also see that at $p_f = 0$, $x = m_3 + m_4$ which when plugged into the differential cross section we get

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{CM} = \frac{\tilde{S}}{(8\pi)^2 E_{\text{tot}}|\mathbf{p}_i|} \int_{m_3+m_4}^\infty p_f(x)|\mathcal{M}_{fi}|^2\delta(E_{\text{tot}} - x)\frac{\mathrm{d}x}{x}, \tag{2.42}$$

where we label $m_3 + m_4$ as the threshold for the reaction. Now in order for $\delta(E_{\text{tot}} - x)$ to have support in the integration region then we must have $E_{tot} > m_3 + m_4$ which leaves

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\tilde{S}}{(8\pi)^2} \frac{p_f^*}{|\mathbf{p}_i|E_{\text{tot}}^2} |\mathcal{M}_{fi}|^2_* \Theta(E_{\text{tot}} - (m_3 + m_4)), \tag{2.43}$$

where the * means it is evaluated at $\mathbf{p}_3 = -\mathbf{p}_4 = -\mathbf{p}_f$ and p_f^* is determined by

$$E_{\text{tot}} \equiv E_T = \sqrt{(p_f^*)^2 + m_3^2} + \sqrt{(p_f^*)^2 + m_4^2} \Leftrightarrow p_f^* = \frac{1}{2E_T}\sqrt{[E_T^2 - (m_3 - m_4)^2][E_T^2 - (m_3 + m_4)^2]}. \tag{2.44}$$

The reaction threshold corresponds to when $E_T = m_3 + m_4 \Rightarrow p_f^* = 0$ i.e. the collision products are at rest in the Center of Mass frame.

2.2 Scalar QED

Start with the Lagrangian for the complex Klein-Gordon Equation

$$\mathcal{L} = -\partial_\mu \Phi^\dagger \partial^\mu \Phi - M^2 \Phi^\dagger \Phi, \tag{2.45}$$

and recall that the theory is invariant under a *global U(1)-gauge symmetry* $\Phi \to e^{i\theta}\Phi$. Next we promote this spacetime constant to a spacetime-varying field $\theta = \theta(\mathbf{r}, t)$. We see that the gauge symmetry has been spoiled

$$\mathcal{L}' = -\partial_\mu \Phi^\dagger \partial^\mu \Phi - \Phi^\dagger \Phi M^2 - i\partial_\mu \theta (\Phi \partial^\mu \Phi^\dagger - \Phi^\dagger \partial^\mu \Phi) - \Phi^\dagger \Phi \partial_\mu \theta \partial^\mu \theta. \tag{2.46}$$

To restore the gauge symmetry, we must find a way to cancel out all the θ terms. Fortunately we know that under a gauge transformation, gauge fields, specifically the 4-potential from E&M, transforms via $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \theta$. Express the new Lagrangian by

$$\mathcal{L} = -(D_\mu \Phi)^\dagger D^\mu \Phi - M^2 \Phi^\dagger \Phi, \tag{2.47}$$

where $D_\mu = \partial_\mu - ieA_\mu$ is the covariant derivative and e is the elementary charge coupling constant. This gives us a new local gauge symmetry for which we want $\mathcal{L}$ to be invariant. To that we write

$$D_\mu \Phi(x) = (\partial_\mu \Phi - ie\Phi A_\mu) \to e^{i\theta(x)} D_\mu \Phi(x), \quad (D_\mu \Phi(x))^\dagger = (\partial_\mu \Phi^\dagger + ie\Phi^\dagger A_\mu) \to (D_\mu \Phi(x))^\dagger e^{-i\theta(x)}, \tag{2.48}$$

and thus the Lagrangian

$$\mathcal{L} = -(D_\mu \Phi)^\dagger (D^\mu \Phi) - M^2 \Phi^\dagger \Phi \tag{2.49}$$

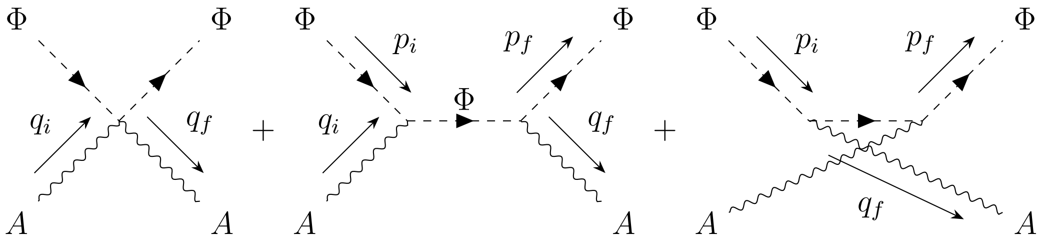
is gauge invariant since $M^2\Phi^\dagger\Phi$ is explicitly gauge invariant. Now that we have a new field A_μ, we can also add its own dynamics i.e. a kinetic term

$$\mathcal{L} = -(\partial_\mu\Phi^\dagger + ieA_\mu\Phi^\dagger)(\partial^\mu\Phi - ieA^\mu\Phi) - M^2\Phi^\dagger\Phi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}. \tag{2.50}$$

Let's calculate the 2→2 scattering process: $\Phi\gamma \to \Phi\gamma$ i.e. Compton scattering.

$\Phi =$ , $A_\mu =$

We are interested in the following (tree-level) diagrams:



The interaction Lagrangian is

$$\mathcal{L}_{int} = -e^2A^\mu A_\mu\Phi^\dagger\Phi - ieA^\mu(\Phi^\dagger\partial_\mu\Phi - \Phi\partial_\mu\Phi^\dagger). \tag{2.51}$$

Let's calculate the amplitude for the first (and easiest) interaction. The initial state is $|i\rangle = |\Phi,\gamma\rangle$ and the final state is $|f\rangle = |\Phi,\gamma\rangle$. For the initial state, we need the operators

$$\hat{\Phi} \to \frac{\hat{a}_{p_i}e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}, \qquad \gamma \to \frac{\hat{a}_{q_i\lambda}\epsilon_i e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}},$$

and the final state is

$$\hat{\Phi}^\dagger \to \frac{\hat{a}^\dagger_{p_f}e^{-ip_f\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}, \qquad \gamma \to \frac{\hat{a}^\dagger_{q_f\lambda}\epsilon_f e^{-iq_f\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}$$

.

Thus the S-matrix element for this process is

$$S_{fi} = (-i)(-e^2)\int \mathrm{d}^4x \left\langle f\middle|\mathcal{T}(\hat{A}^\mu \hat{A}_\mu \hat{\Phi}^\dagger \hat{\Phi})\middle|i\right\rangle \tag{2.52}$$
$$= ie^2 \int \mathrm{d}^4x \,\frac{e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}\frac{e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}}\frac{e^{-ip_f\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}\frac{e^{-iq_f\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}\epsilon_i^\mu \epsilon_{\mu,f} \tag{2.53}$$
$$= \frac{ie^2\epsilon_i\cdot\epsilon_f}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}2\mathcal{V}\omega_{\mathbf{q}_i}2\mathcal{V}E_{\mathbf{p}_f}2\mathcal{V}\omega_{\mathbf{q}_f}}}\int \mathrm{d}^4x\, e^{i(p_i+q_i-p_f-q_f)\cdot x} \tag{2.54}$$
$$= \frac{(2\pi)^4\delta^{(4)}(p_i+q_i-p_f-q_f)\mathcal{M}_{fi}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}2\mathcal{V}\omega_{\mathbf{q}_i}2\mathcal{V}E_{\mathbf{p}_f}2\mathcal{V}\omega_{\mathbf{q}_f}}}, \tag{2.55}$$

where $\mathcal{M}_{fi} = ie^2\epsilon_i\cdot\epsilon_f$. There is also the diagram where we swap the legs for the photon external lines but since that amplitude is identical to this one, the total contribution to the amplitude is $i\mathcal{M}_{fi} = -2e^2\epsilon_i\cdot\epsilon_f$. Now we can focus on the second interaction term, however this will require much more care. The S-matrix element looks like

$$S_{fi} = \frac{(-i)^2(-ie)^2}{2!}\int \mathrm{d}^4x \int \mathrm{d}^4y\, A^\mu(x)A^\nu(y)[\Phi^\dagger(x)\partial_\mu\Phi(x) - \Phi(x)\partial_\mu\Phi^\dagger(x)][\Phi^\dagger(y)\partial_\nu\Phi(y) - \Phi(y)\partial_\nu\Phi^\dagger(y)]. \tag{2.56}$$

We can get rid of the factor of 2 in the denominator since this integral is symmetric in x, y. Now because the initial and final state are the same in the previous diagram, we require the use of the same operators

$$\hat{\Phi} \to \frac{\hat{a}_{p_i}e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}},\quad \gamma \to \frac{\hat{a}_{q_i\lambda}\epsilon_i e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}},\quad \hat{\Phi}^\dagger \to \frac{\hat{a}^\dagger_{p_f}e^{-ip_f\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}},\quad \gamma \to \frac{\hat{a}^\dagger_{q_f\lambda}\epsilon_f e^{-iq_f\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}$$

.

The S-matrix is then

$$\begin{aligned} S_{fi} = e^2\int \mathrm{d}^4x\int \mathrm{d}^4y \Bigg[&\left(\frac{e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}\right)\left(\frac{\epsilon_i^\mu e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}}\right)\left(\frac{(-ip_{\nu,f})e^{-ip_f\cdot y}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}\right)\left(\frac{\epsilon_f^\nu e^{-iq_f\cdot y}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}\right)\left\langle 0\middle|\mathcal{T}(\partial_\mu\Phi^\dagger\Phi)\middle|0\right\rangle \\ &-\left(\frac{e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}\right)\left(\frac{\epsilon_i^\mu e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}}\right)\left(\frac{e^{-ip_f\cdot y}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}\right)\left(\frac{\epsilon_f^\nu e^{-iq_f\cdot y}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}\right)\left\langle 0\middle|\mathcal{T}(\partial_\mu\Phi^\dagger\partial_\nu\Phi)\middle|0\right\rangle \\ &-\left(\frac{(ip_{\mu,i})e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}\right)\left(\frac{\epsilon_i^\mu e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}}\right)\left(\frac{(-ip_{\nu,f})e^{-ip_f\cdot y}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}\right)\left(\frac{\epsilon_f^\nu e^{-iq_f\cdot y}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}\right)\left\langle 0\middle|\mathcal{T}(\Phi^\dagger(x)\Phi(y))\middle|0\right\rangle \\ &+\left(\frac{(ip_{\mu,i})e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}\right)\left(\frac{\epsilon_i^\mu e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}}\right)\left(\frac{e^{-ip_f\cdot y}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}\right)\left(\frac{\epsilon_f^\nu e^{-iq_f\cdot y}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}\right)\left\langle 0\middle|\mathcal{T}(\Phi^\dagger\partial_\nu\Phi)\middle|0\right\rangle\Bigg], \end{aligned} \tag{2.57}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \partial_\nu \equiv \frac{\partial}{\partial y^\nu}$. Now we need to make use of the following definitions

$$\langle 0 \big| \mathcal{T}(\Phi(x)\Phi^\dagger(y)) \big| 0 \rangle = i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{-k^2 - M^2 + i\varepsilon}, \tag{2.58}$$

$$\langle 0 \big| \mathcal{T}(\partial_\mu\Phi(x)\Phi^\dagger(y)) \big| 0 \rangle = - \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\mu e^{ik\cdot(x-y)}}{-k^2 - M^2 + i\varepsilon}, \tag{2.59}$$

$$\langle 0 \big| \mathcal{T}(\Phi(x)\partial_\nu\Phi^\dagger(y)) \big| 0 \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\nu e^{ik\cdot(x-y)}}{-k^2 - M^2 + i\varepsilon}, \tag{2.60}$$

$$\langle 0 \big| \mathcal{T}(\partial_\mu\Phi(x)\partial_\nu\Phi^\dagger(y)) \big| 0 \rangle = i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{k_\mu k_\nu e^{ik\cdot(x-y)}}{-k^2 - M^2 + i\varepsilon}, \tag{2.61}$$

to plug them into the S-matrix element and we find

$$\begin{gathered} S_{fi} = \frac{e^2}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i} 2\mathcal{V}\omega_{\mathbf{q}_i} 2\mathcal{V}E_{\mathbf{p}_f} 2\mathcal{V}\omega_{\mathbf{q}_f}}} \int \mathrm{d}^4 x \int \mathrm{d}^4 y \, e^{i[(p_i+q_i)\cdot x - (p_f+q_f)\cdot y]} \\ \Bigg[-i(ip_i\cdot\epsilon_i)(ip_f\cdot\epsilon_f) \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \frac{e^{-ik_1\cdot(x-y)}}{-k_1^2 - M^2 - i\varepsilon} + (ip_f\cdot\epsilon_f)\epsilon_i^\mu \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{k_{\mu,2} e^{-ik_2\cdot(x-y)}}{-k_2^2 - M^2 - i\varepsilon} \\ + (ip_i\cdot\epsilon_i)\epsilon_f^\nu \int \frac{\mathrm{d}^4 k_3}{(2\pi)^4} \frac{k_{\nu,3} e^{-ik_3\cdot(x-y)}}{-k_3^2 - M^2 - i\varepsilon} + i\epsilon_i^\mu\epsilon_f^\nu \int \frac{\mathrm{d}^4 k_4}{(2\pi)^4} \frac{k_{\mu,4} k_{\nu,4} e^{-ik_4\cdot(x-y)}}{-k_4^2 - M^2 - i\varepsilon} \Bigg], \end{gathered} \tag{2.62}$$

which simplifies to be

$$\begin{aligned} S_{fi} = \mathcal{E} \Bigg[& i(p_i\cdot\epsilon_i)(p_f\cdot\epsilon_f) \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 k_1}{(2\pi)} \frac{e^{i(p_i+q_i-k_1)\cdot x} e^{i(k_1-p_f-q_f)\cdot y}}{-k_1^2 - M^2 - i\varepsilon} \\ & + i(p_f\cdot\epsilon_f)\epsilon_i^\mu \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{k_{\mu,2} e^{i(p_i+q_i-k_2)\cdot x} e^{i(k_2-p_f-q_f)\cdot y}}{-k_2^2 - M^2 - i\varepsilon} \\ & + i(p_i\cdot\epsilon_i)\epsilon_f^\nu \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 k_3}{(2\pi)^4} \frac{k_{\nu,3} e^{i(p_i+q_i-k_3)\cdot x} e^{i(k_3-p_f-q_f)\cdot y}}{-k_3^2 - M^2 - i\varepsilon} \\ & + i\epsilon_i^\mu\epsilon_f^\nu \int \mathrm{d}^4 x \int \mathrm{d}^4 y \int \frac{\mathrm{d}^4 k_4}{(2\pi)^4} \frac{k_{\mu,4} k_{\nu,4} e^{i(p_i+q_i-k_4)\cdot x} e^{i(k_4-p_f-q_f)\cdot y}}{-k_4^2 - M^2 - i\varepsilon} \Bigg], \end{aligned} \tag{2.63}$$

where $\mathcal{E} = \frac{e^2}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i} 2\mathcal{V}\omega_{\mathbf{q}_i} 2\mathcal{V}E_{\mathbf{p}_f} 2\mathcal{V}\omega_{\mathbf{q}_f}}}$. Now we're ready to carry out the integrals. We can integrate with respect to x for each term and they all will yield $\int \mathrm{d}^4 x \to (2\pi)^4 \delta^{(4)}(p_i + q_i - k_a)$ similarly with integrating with respect to y $\int \mathrm{d}^4 y \to (2\pi)^4 \delta^{(4)}(k_a - p_f - q_f)$ where $a = 1, \ldots, 4$. Thus when we integrate with respect to k_a, those integrals all yield $\int \mathrm{d}^4 k_a \to (2\pi)^4 \delta^{(4)}(p_i + q_i - p_f - q_f)$. Thus the amplitude becomes

$$i\mathcal{M}^{I}_{fi} = -\frac{4e^2(p_i\cdot\epsilon_i)(p_f\cdot\epsilon_f)}{-(p_f+q_f)^2 - M^2}, \tag{2.64}$$

where we used the fact that the Dirac delta function enforces momentum conservation so $p_i + q_i = p_f + q_f$ as well as the fact that the photon is transverse so $\epsilon_i \cdot q_i = \epsilon_f \cdot q_f = 0$. Further simplifications can be made by recognizing that $p^2 = -M^2$ and $q^2 = 0$ which leaves

$$i\mathcal{M}^{I}_{fi} = \frac{2e^2(p_i\cdot\epsilon_i)(p_f\cdot\epsilon_f)}{p_i\cdot q_i}. \tag{2.65}$$

We can find the amplitude for the last diagram by making the observation that it'll be the same as this one by making the substitution $q_i \leftrightarrow -q_f$ which gives

$$i\mathcal{M}^{II}_{fi} = -\frac{2e^2(p_i\cdot\epsilon_i)(p_f\cdot\epsilon_f)}{p_i\cdot q_f}. \tag{2.66}$$

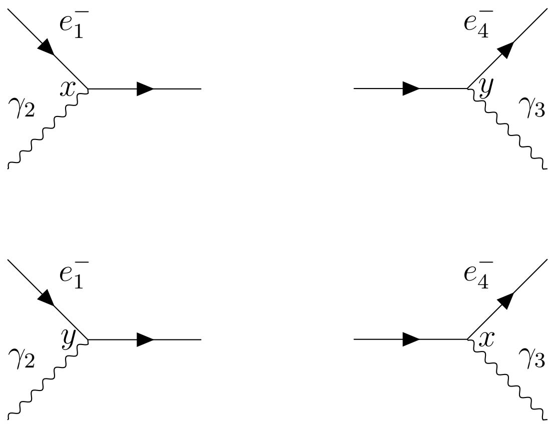
Meaning the total tree-level amplitude for this process is

$$i\mathcal{M}^{tot}_{fi} = 2e^2\left[\frac{(p_i\cdot\epsilon_i)(p_f\cdot\epsilon_f)}{p_i\cdot q_i} - \frac{(p_i\cdot\epsilon_i)(p_f\cdot\epsilon_f)}{p_i\cdot q_f} - \epsilon_i\cdot\epsilon_f\right] \tag{2.67}$$

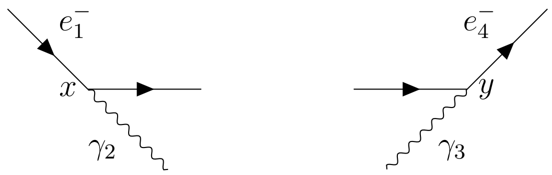
2.3 Compton Scattering

$$e^-\gamma \rightarrow e^-\gamma$$

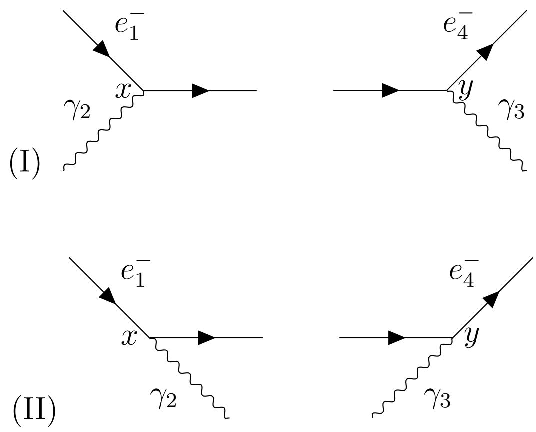
Here our initial state $|i\rangle = \left|e_1^-, \gamma_2\right\rangle$ and our final state $|f\rangle = \left|e_4^-, \gamma_3\right\rangle$. So we wish to destroy γ_2 and create γ_3 which means we need two A's which implies we need to go second order. At x, we destroy e_1^- and create e_4^- and the same can be done at y



Notice how these diagrams are exactly the same by $x \leftrightarrow y \Rightarrow \frac{1}{2!}$. However at x, we can either destroy γ_2 or create γ_3. In the latter case γ_2 is destroyed at y



Thus we have two independent contributions to the scattering amplitude



The "dangling" fermion lines will "join" into an internal fermion propagator. This is seen as follows: in second order we have the term $\overline{\psi}(y)\not{A}(y)\psi(y)\overline{\psi}(x)\not{A}(x)\psi(x)$. The $\psi(x) \sim \hat{b}+\hat{d}^\dagger$ so use $\hat{b}$ to "kill" e_1 at x. At y, $\overline{\psi}(y) \sim \hat{b}^\dagger + \hat{d}$ so use $\hat{b}^\dagger$ to create e_4 or acting on $\langle f| : \langle e_4| \, \hat{b}_4^\dagger = \langle 0| \Rightarrow \psi(x) \Rightarrow \hat{b}_1 \, |e_1\rangle = 0$. There remains $\langle 0|\psi(x)\overline{\psi}(y)|0\rangle$ (and the A's create/destroy the photons in $|f\rangle \, , |i\rangle$) with the $\mathcal{T}$ symbol. Thus we have the fermion propagator. The second order terms need

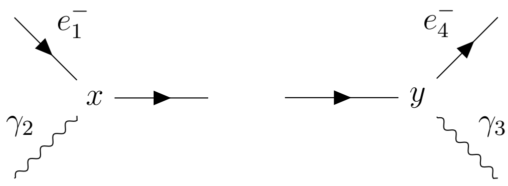
$$\langle f|\mathcal{T}\big(\overline{\psi}_C(y)\gamma^\mu_{CD}\psi_D(y)\overline{\psi}_A(x)\gamma^\nu_{AB}\psi_B(x)\big)A_\mu(y)A_\nu(x)|i\rangle \, , \tag{2.68}$$

where we carefully kept the Dirac indices. Writing the $\mathcal{T}$-symbol explicitly there are two terms

$$(a)\,\langle e_4, \gamma_3|\big(\overline{\psi}_C(y)\gamma^\mu_{cd}\psi_D(y)\overline{\psi}_A(x)\gamma^\nu_{AB}\psi_B(x)A_\mu(y)A_\nu(x)\Theta(\tau - t)\big)|e_1, \gamma_2\rangle \, , \tag{2.69}$$

$$(b)\,\langle e_4, \gamma_3|\big(\overline{\psi}_A(x)\gamma^\mu_{AB}\psi_B(x)\overline{\psi}_C(x)\gamma^\nu_{CD}\psi_D(x)A_\mu(x)A_\nu(y)\Theta(t - \tau)\big)|e_1, \gamma\rangle \, , \tag{2.70}$$

and consider



In (a): use $\hat{b}^\dagger$ from $\overline{\psi}_C(y)$: $\langle e_4|\,\hat{b}^\dagger = \langle 0|$, and $\hat{b}$ from $\psi_B(x)$ such that $\hat{b}_1\,|e_1\rangle = 0 \Rightarrow$ (don't write normalization and exponential factors) and get

$$(a) :\ \langle 0\gamma_3|\bar{u}_c(4)\gamma^\mu_{CD}\psi_D(y)\overline{\psi}_A(x)\gamma^\nu_{AB}u_B(1)A_\mu(y)A_\nu(x)\Theta(\tau - t)|0\gamma_2\rangle\,. \tag{2.71}$$

Now we can use $A_\nu(x)$ ($\hat{a}$- term) to destroy γ_2 at x and the $\hat{a}^\dagger$ term from $A_\mu(y)$ to create γ_3 at y. There remains the expectation value

$$\langle 0|\psi_D(y)\overline{\psi}_A(x)|0\rangle\,\Theta(\tau - t). \tag{2.72}$$

In (b): use $\hat{b}_1$ from $\psi(x)$: $\hat{b}_1\,|e_1\rangle = |0\rangle$ and $\hat{b}_4^\dagger$ from $\overline{\psi}(y)$: $\langle e_4|\,\hat{b}_4^\dagger = \langle 0|$ where as in (a) the ordering was $\hat{b}_4^\dagger\hat{b}_1$ now for (b), the ordering is $\hat{b}_1\hat{b}_4^\dagger \to -\hat{b}_4^\dagger\hat{b}_1$ which implies there is a relative sign $\Rightarrow$ in (b), the remaining Fermi operators are

$$-\,\langle 0|\overline{\psi}_A(x)\psi_D(y)|0\rangle\,\Theta(t - \tau), \tag{2.73}$$

and use $A_\mu A_\nu$ to create/annihilate the photons as in (a). There remains

$$\langle 0|\psi_D(y)\overline{\psi}_A(x)|0\rangle\,\Theta(\tau - t) - \langle 0|\overline{\psi}_A(x)\psi_D(y)|0\rangle\,\Theta(t - \tau) \equiv \langle 0|\mathcal{T}\big(\psi_D(y)\overline{\psi}_A(x)\big)|0\rangle\,, \tag{2.74}$$

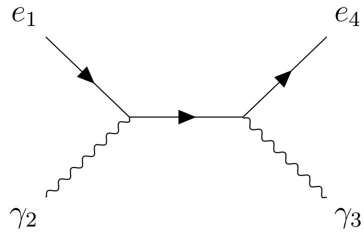
which is the fermion propagator given by

$$\langle 0|\mathcal{T}\big(\psi_D(y)\overline{\psi}_A(x)\big)|0\rangle = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{(\not{k} + m_e)_{DA}}{k^2 - m_e^2 + i\epsilon}e^{ik\cdot(x-y)}. \tag{2.75}$$

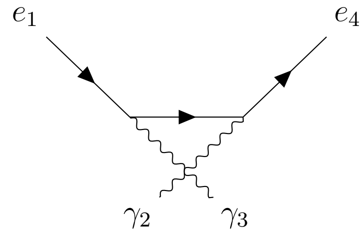
Now we put everything together to get

Diagram II differs from I by "swapping" the external bosonic lines and labels 2↔3

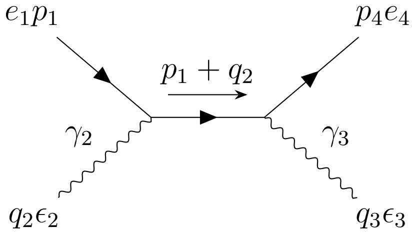
with $A_\mu \sim (\hat{a} + \hat{a}^\dagger)\epsilon_\mu$ after using $\hat{a}, \hat{a}^\dagger$ the $\epsilon-$polarization vectors remain $\epsilon^\mu = (0, \epsilon^i)$ at $x \Rightarrow$ that $k = p_1 + q_2$ and at $y \Rightarrow k = q_3 + p_4$. Thus the scattering amplitude for this diagram is



(I)



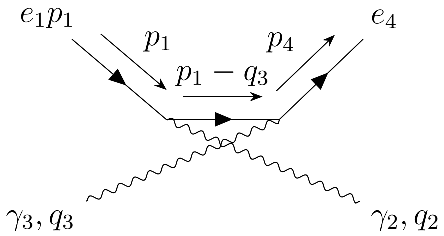
(II)



(I)

$$\mathcal{M}^{I}_{fi} = (-ie)^2 \bar{u}_C(4)\gamma^{\mu}_{CD}\epsilon_{\mu}(3)\frac{i(\not{k}+m_e)_{DA}}{k^2 - m_e^2 + i\epsilon}\gamma^{\nu}_{AB}\epsilon_{\nu}(2)u_B(1). \tag{2.76}$$

For II, create γ_3 at $x \Rightarrow \epsilon_{\nu}(3)e^{iq_3\cdot x}$ and destroy $\gamma_2 \Rightarrow \epsilon_{\mu}(2)e^{-iq_2\cdot y}$



(II)

From I, "swap" $\epsilon_{\mu}(3) \leftrightarrow \epsilon_{\mu}(2), \epsilon_{\nu}(2) \leftrightarrow \epsilon_{\nu}(3)$ and $q_3 \to -q_2,\ q_2 \to -q_3$. This is because now at x, q_3 goes out of the vertex $(e^{iq_3\cdot x})$ and q_2 goes into the vertex at $y(e^{-iq_2\cdot y})$ which leads to the scattering amplitude

$$\mathcal{M}^{(II)}_{fi} = (-ie)^2 \bar{u}_C(4)\gamma^{\mu}_{CD}\epsilon_{\mu}(2)\frac{i(\not{k}+m_e)_{DA}}{k^2 - m_e^2 + i\epsilon}\gamma^{\nu}_{AB}\epsilon_{\nu}(3)u_B(1), \tag{2.77}$$

where $k = p_1 - q_3 = p_4 - q_2 \Rightarrow p_1 + q_2 = q_3 + p_4$. What is the relative sign? Swapping $\gamma_2 \leftrightarrow \gamma_3 \Rightarrow$ swapped only photon lines $\hat{a}_2\hat{a}_3^{\dagger} = \hat{a}_3^{\dagger}\hat{a}_2$ (commute for 2≠3). Thus there's a relative + sign. We can see this by noticing that swapping the electron lines also swap the lines in the Fermi propagator ⇒ swapping all 4-Fermi lines⇒ they commute⇒ relative + sign⇒ $\mathcal{M}^{tot}_{fi} = \mathcal{M}^{(I)}_{fi} + \mathcal{M}^{(II)}_{fi}$. Thus the scattering amplitude for Compton Scattering is

$$\mathcal{M}_{fi}^{tot} = i(-ie)^2 \left[\bar{u}(4)\not\epsilon(3) \frac{\not p_1 + \not q_2 + m_e}{(p_1+q_2)^2 - m_e^2 + i\epsilon} \not\epsilon(2) u(1) + \bar{u}(4)\not\epsilon(2) \frac{\not p_1 - \not q_3 + m_e}{(p_1 - q_3)^2 - m_e^2 + i\epsilon} \not\epsilon(3) u(1) \right]. \tag{2.78}$$

2.3.1 Feynman Rules

(1) Draw the different Feynman Diagrams: relabeling $x \leftrightarrow y$ cancels $\frac{1}{2!}$

(2) For each incoming fermion line: $\hat{b} \to u$ and for each outgoing fermion line $\hat{b}^\dagger \to \bar{u}$. For incoming anti-fermions $\hat{d} \to \bar{v}$ and for outgoing anti-fermions $\hat{d}^\dagger \to v$. Arrange the spinors in order $(\bar{u}, \bar{v})\gamma(u, v)$ from $\overline{\psi}\gamma^\mu\psi$.

(3) Conserve energy/momentum at each vertex: the transfer momentum to an internal propagator is $p_{in} - p_{out} \Rightarrow$ total E/p conserved.

(4) In-out photons ("on-shell") $\Rightarrow \epsilon^\mu = (0, \hat{\epsilon})$.

(5) Internal scalar propagator:

$$\frac{i}{k^2 - \mu^2 + i\epsilon},$$

Internal photon propagator:

$$\frac{-ig_{\mu\nu}}{k^2 + i\epsilon},$$

Internal MVB propagator:

$$\frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right)}{k^2 - M^2 + i\epsilon},$$

Internal fermion propagator:

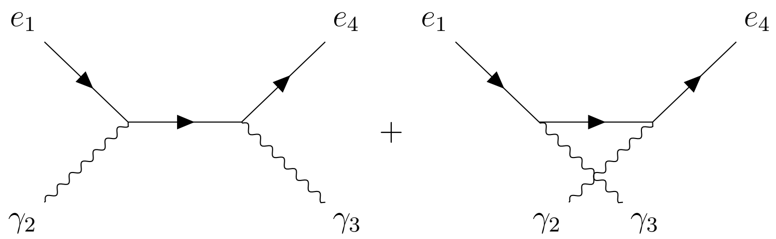
$$\frac{i(\not k + m)}{k^2 - m^2 + i\epsilon},$$

(6) "Swapping" external bosonic lines $\Rightarrow$ relative (+) sign and swapping external fermion lines $\Rightarrow$ relative (-) sign (from anti-commutators)

(7) In-out particles obey free-field dispersion relations "on-shell" $k^2 = M^2$

2.3.2 Compton Scattering Revisited: Gauge Invariance and Ward Identities

Recall the diagrams and $\mathcal{M}_{fi}$ for $e_1\gamma_2 \to e_4\gamma_3$



$$\mathcal{M}_{fi} = i(-ie)^2 \left[\bar{u}(4)\not\epsilon(3) \frac{\not p_1 + \not q_2 + m_e}{(p_1+q_2)^2 - m_e^2 + i\epsilon} \not\epsilon(2) u(1) + \bar{u}(4)\not\epsilon(2) \frac{\not p_1 - \not q_3 + m_e}{(p_1-q_3)^2 - m_e^2 + i\epsilon} \not\epsilon(3) u(1) \right]. \quad (2.79)$$

The photons are external on-shell with $q_2^2 = q_3^2 = 0$. We can write

$$\mathcal{M}_{fi} = \mathcal{M}_{\mu\nu}(p_1, p_4; q_2, q_3)\epsilon^\mu(q_2)\epsilon^\nu(q_3), \quad (2.80)$$

where

$$\mathcal{M}_{\mu\nu} = -ie^2 \left[\bar{u}(4)\gamma^\mu \frac{1}{\not p_1 + \not q_2 - m_e} \gamma^\nu u(1) + \bar{u}(4)\gamma^\nu \frac{1}{\not p_1 - \not q_3 - m_e} \gamma^\mu u(1) \right], \quad (2.81)$$

with $\frac{1}{\not Q - m} = \frac{\not Q + m}{Q^2 - m^2}$. This result is more general: for an S-matrix element with external (on-shell) photons, the $\mathcal{M}_{fi}$ can always be written as $\mathcal{M}_{\mu\nu\alpha\beta\ldots}\epsilon^\mu(p_1)\epsilon^\nu(p_2)\epsilon^\alpha(p_3)\cdots$

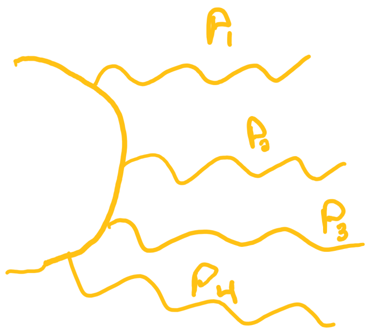


Figure 16: A source emitting photons. The S-matrix must maintain its gauge invariance in spite of the use of gauge fields here.

The photons are physical, transverse ($\epsilon^\mu_\lambda = (0, \hat{\epsilon})$) but the S_{fi} must be gauge invariant with

$$\hat{A}^\mu(x) = \frac{1}{\sqrt{\mathcal{V}}} \sum_{\mathbf{p},\lambda} \frac{\epsilon^\mu_\lambda(p)}{\sqrt{2E_\mathbf{p}}} \left[\hat{a}_{\mathbf{p},\lambda} e^{-ip\cdot x} + \hat{a}^\dagger_{\mathbf{p},\lambda} e^{ip\cdot x} \right]. \quad (2.82)$$

Under a gauge transformation: $A^\mu \to A^\mu + \partial^\mu \Lambda \Rightarrow \epsilon^\mu_\lambda(p) \to \epsilon^\mu_\lambda(p) + p^\mu \tilde{\Lambda}$. Therefore the gauge invariance of S_{fi} must imply

$$p_1^\mu \mathcal{M}_{\mu\nu\alpha\beta} = p_2^\nu \mathcal{M}_{\mu\nu\alpha\beta} = p_3^\alpha \mathcal{M}_{\mu\nu\alpha\beta} \cdots = 0. \quad (2.83)$$

These are Ward Identities which are a consequence of Gauge Invariance. Consider $p_3^\mu \mathcal{M}_{\mu\nu} = 0$. Total E/p conservation means $p_1 + q_2 = p_4 + q_3$ which also means

$$\frac{1}{\not p_1 + \not q_2 - m_e} = \frac{1}{\not p_4 + \not q_3 - m_e}. \quad (2.84)$$

When we put in current conservation at each vertex

$$q_3^\mu \mathcal{M}_{\mu\nu} = -ie^2\left[\bar{u}(4)\not{q}_3 \frac{1}{\not{p}_4 - \not{q}_3 - m_e}\gamma^\nu u(1) + \bar{u}(4)\frac{1}{\not{p}_1 - \not{q}_3 - m_e}\not{q}_3 u(1)\right]. \tag{2.85}$$

Writing $\not{q}_3 = \not{p}_4 + \not{q}_3 - m_e - (\not{p}_4 - m_e)$ and $\bar{u}(4)(\not{p}_4 - m_e) = 0$. So the first term becomes $\bar{u}(4)\gamma^\nu u(1)$. The second term $\not{q}_3 = -(\not{p}_1 - \not{q}_3 - m_e) + (\not{p}_1 - m_e)$ and $(\not{p}_1 - m_e)u(1) = 0$ which means the second term becomes $-\bar{u}(4)\gamma^\nu u(1)$. So $q_3^\mu \mathcal{M}_{\mu\nu} = 0$. Note that the individual terms in $\mathcal{M}_{\mu\nu}$ do not cancel. Only the sum of all contributions cancel. This enforces S_{fi} is gauge invariant but all Feynman diagrams at a given order must be included. Same for $q_2^\mu \mathcal{M}_{\mu\nu} = 0$.

The Ward Identities simplify the calculation of unpolarization $|\mathcal{M}_{fi}|^2$ by summing over the final photon polarizations. To show the main aspect, consider only one photon in the final state $\mathcal{M}_{fi} = \epsilon_\lambda^\mu(p)\mathcal{M}_\mu$:

$$|\mathcal{M}_{fi}|^2 = \mathcal{M}_\mu^* \mathcal{M}_\nu \epsilon_\lambda^\mu \epsilon_\lambda^\nu. \tag{2.86}$$

The physical photons carry the constraint $q^2 = 0$ with two transverse polarizations. The summed over polarizations yields

$$\sum_{\lambda=1}^{2} |\mathcal{M}_{fi}|^2 = \mathcal{M}_\mu^* \mathcal{M}_\nu \sum_{\lambda=1}^{2} \epsilon_\lambda^\mu \epsilon_\lambda^\nu. \tag{2.87}$$

Define $\eta^\mu = (1, 0, 0, 0); q \cdot \eta = \omega(q)$. Using

$$\sum_{\lambda=1}^{2} \epsilon_\lambda^\mu \epsilon_\lambda^\nu = -g^{\mu\nu} - \frac{q^\mu q^\nu}{(q\cdot\eta)^2} + \frac{(q\cdot\eta)}{(q\cdot\eta)^2}(q^\mu \eta^\nu + q^\nu \eta^\mu). \tag{2.88}$$

Because $q^\mu \mathcal{M}_\mu = 0$, only the $-g^{\mu\nu}$ term contributes therefore

$$\sum_{\lambda=1}^{2} |\mathcal{M}_{fi}|^2 = -g^{\mu\nu}\mathcal{M}_\mu^* \mathcal{M}_\nu. \tag{2.89}$$

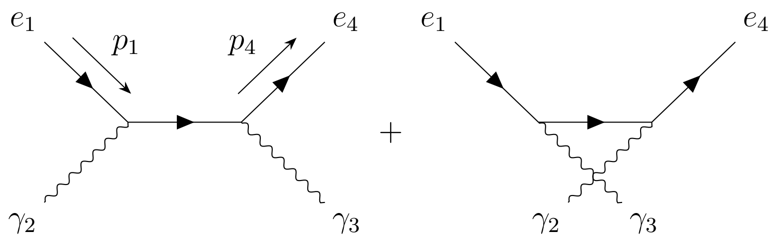
For unpolarized Compton Scattering

$$\sum_{\lambda=1}^{2}\sum_{\rho=1}^{2} \mathcal{M}_{\mu\nu}^* \mathcal{M}_{\alpha\beta} \epsilon_\lambda^\mu(q_2)\epsilon_\lambda^\nu(q_2)\epsilon_\rho^\alpha(q_3)\epsilon_\rho^\beta(q_3) = g^{\mu\alpha}g^{\nu\beta}\mathcal{M}_{\mu\nu}^*\mathcal{M}_{\alpha\beta}. \tag{2.90}$$

2.3.3 Compton Scattering Cross Section: Klein-Nishina + Thompson

Carrying out the traces, but without summing over the photon polarizations gives

$$\overline{|\mathcal{M}_{fi}|^2} = e^4 \left[\frac{q' \cdot p}{q \cdot p} + \frac{q \cdot p}{q' \cdot p} + 4\epsilon(q) \cdot \epsilon'(q') - 2 \right], \tag{2.91}$$



where $q^\mu = (\omega, \mathbf{q}), q'^\mu = (\omega', \mathbf{q}')$. In the lab frame where the initial electron is at rest, $p^\mu = (m_e, 0)$ and we get the Klein-Nishina Formula

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\mathrm{lab}} = \frac{\alpha_{\mathrm{EM}}^2}{4m_e^2} \left[\frac{\omega'}{\omega}\right]^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4(\epsilon \cdot \epsilon') - 2 \right], \tag{2.92}$$

where E/p conservation yields

$$\frac{\omega'}{\omega} = \frac{1}{1 + \frac{\omega}{m_e}(1 - \cos\theta)}. \tag{2.93}$$

In the low energy limit $\omega'/\omega \simeq 1$

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\mathrm{lab}} = \frac{\alpha_{\mathrm{EM}}^2}{4m_e^2} \cdot 4(\epsilon \cdot \epsilon')^2. \tag{2.94}$$

Averaging over initial and summing over final photon polarizations

$$\frac{1}{2}\sum_{\lambda}\sum_{\lambda'}(\epsilon \cdot \epsilon')^2 = \frac{1}{2}(1 + \cos^2\theta) = 1 - \frac{1}{2}\sin^2\theta, \tag{2.95}$$

which when applied to the differential cross section gives

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right|_{\mathrm{lab}} = \frac{\alpha_{\mathrm{EM}}^2}{2m_e^2}(1 + \cos^2\theta). \tag{2.96}$$

2.3.4 Differential Thompson Scattering Cross Section

The Thompson Cross Section is

$$\sigma_{\rm Th} = \int \frac{{\rm d}\sigma}{{\rm d}\Omega}\,{\rm d}\Omega = \frac{8\pi}{3}\frac{\alpha_{\rm EM}^2}{m_e^2} \simeq 0.665\times 10^{-24}\,{\rm cm}^2. \tag{2.97}$$

This can be calculated from the full Klein-Nishina differential cross section (and using $\frac{1}{2}\sum_\lambda\sum_{\lambda'}(\epsilon\cdot\epsilon')^2$) which is

$$\left.\frac{{\rm d}\sigma}{{\rm d}\Omega}\right|_{\rm lab} = \frac{\alpha_{\rm EM}^2}{2m_e^2}\left[\frac{\omega'}{\omega}\right]^2\left[\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2\theta\right]. \tag{2.98}$$

In the low energy limit we recover Thompson Scattering but in the high energy limit $\frac{\omega'}{\omega}\sim\frac{m_e}{\omega}\frac{1}{1-\cos\theta}$ and features a divergence as $\theta\to 0$. This is a collinear singularity: the intermediate propagator is $\sim\frac{1}{(p_1+q_2)^2-m^2}$ and $(p_1+q_2)^2 = p_1^2+q_2^2+2p_1\cdot q_2 = (p_4+q_3)^2 \Rightarrow (p_4+q_3)^2 - m_e^2 = 2p_4\cdot q_3 = 2(E_4\omega_4 - \mathbf{p}_4\cdot\mathbf{q}_3)$ with $\omega_3 = |\mathbf{q}_3|$. In the high energy limit $E_4\sim|\mathbf{p}_4| \Rightarrow 2|\mathbf{p}_3||\mathbf{p}_4|(1-\cos\theta)$. This collinear singularity in the high energy limit is

$$\sigma_{\rm KN} = \frac{2\pi\alpha_{\rm EM}^2}{s}\ln\left(\frac{s}{m_e^2}\right). \tag{2.99}$$

In the low energy limit, the Thompson Scattering cross section features the quantity $r_0 = \frac{\alpha_{\rm EM}}{m_e} = 2.82\times 10^{-13}\,{\rm cm}$ which is called the "charge radius of the electron". This corresponds to the radius r_0 such that the Coulomb-"self energy" of the electron

$$\frac{e^2}{4\pi r_0} = m_e c^2, \tag{2.100}$$

which is the rest energy and

$$\sigma_{\rm Th} = \frac{8\pi r_0^2}{3}. \tag{2.101}$$

This represents one of the largest cross sections in particle physics. It is almost as big as nuclear cross sections. Thompson scattering is the primary process that establishes local thermodynamic equilibrium in the electron photon plasma in the early universe and establishes the temperature of the CMB. The term $(\epsilon\cdot\epsilon')^2\to\sin^2\theta$ in $\frac{{\rm d}\sigma}{{\rm d}\Omega}$ leads to a quadrupole polarization of the photon distribution.

2.3.5 Lessons Learned from Compton Scattering

(1) Ward Identities: Gauge invariance imposes constraints on the $\mathcal{M}_{fi}$ and is only fulfilled by summing of all diagrams at a given order.

(2) Klein-Nishina Cross Sections: Thompson scattering is the low energy limit with $r_0^2 \sim 10^{-24}\,\text{cm}^2$ where r_0 is the classical charge radius of the electron $\sim 10^{-13}\,\text{cm}$.

(3) Collinear singularities: In the high energy limit, collinear singularities lead to log enhancement of cross sections.

3 Gravitational Compton Scattering

3.1 Graviton-Complex Scalar Scatter

We start with the Lagrangian

$$\sqrt{-g}\mathcal{L} = \frac{1}{16\pi G}\sqrt{-g}R - \sqrt{-g}\big(g^{\mu\nu}\nabla_\mu\Phi^\dagger\nabla_\nu\Phi + M^2\Phi^\dagger\Phi\big). \tag{3.1}$$

We shall be expanding our flat space $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ where $\kappa^2 = 32\pi G$. From here we can expand the inverse and volume element

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \ldots, \qquad \sqrt{-g} = 1 + \frac{\kappa}{2}h + \frac{\kappa^2}{8}(h^2 - 2h^{\alpha\beta}h_{\alpha\beta}) + \ldots, \tag{3.2}$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. First we'll pay attention to the Lagrangian for the scalar field $\mathcal{L}_\Phi$. We shall also ignore terms that go like h or $\partial^\mu h_{\mu\nu}$ since the polarization tensors are both traceless and transverse. The Lagrangian becomes

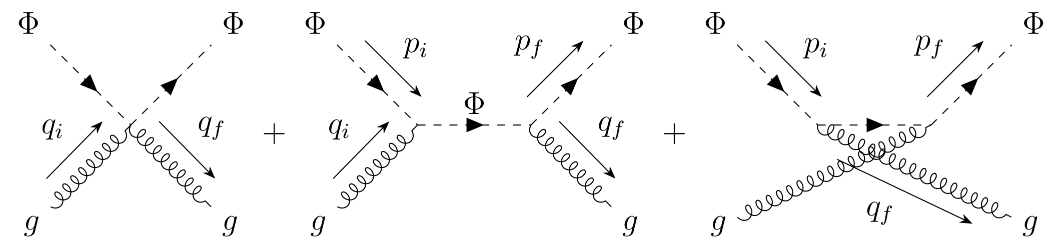
$$\mathcal{L}_\Phi = -\left(1 - \frac{\kappa^2}{4}(h_{\alpha\beta})^2\right)\big[(\eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\lambda}h^\nu_\lambda + \ldots)\partial_\mu\Phi^\dagger\partial_\nu\Phi + M^2\Phi^\dagger\Phi\big] \tag{3.3}$$

$$= -(\partial^\mu\Phi^\dagger\partial_\mu\Phi + M^2\Phi^\dagger\Phi) + \kappa h^{\mu\nu}\partial_\mu\Phi^\dagger\partial_\nu\Phi - \kappa^2 h^{\mu\lambda}h^\nu_\lambda\partial_\mu\Phi^\dagger\partial_\nu\Phi + \frac{\kappa^2}{4}(h_{\lambda\rho})^2(\partial^\mu\Phi^\dagger\partial_\mu\Phi + M^2\Phi^\dagger\Phi). \tag{3.4}$$

The interaction Lagrangian is

$$\mathcal{L}_{int} = \kappa h^{\mu\nu}\partial_\mu\Phi^\dagger\partial_\nu\Phi - \kappa^2 h^{\mu\lambda}h^\nu_\lambda\partial_\mu\Phi^\dagger\partial_\nu\Phi + \frac{\kappa^2}{4}(h_{\lambda\rho})^2(\partial^\mu\Phi^\dagger\partial_\mu\Phi + M^2\Phi^\dagger\Phi) \equiv \mathcal{L}_{gg\Phi} + \mathcal{L}_{gg\Phi\Phi}. \tag{3.5}$$

The Feynman diagrams for these interactions are



The S-matrix elements can be found via the normal processes

$$S_{fi} = (-i)\big(-\kappa^2\big) \int \mathrm{d}^4x \left\langle f \middle| \mathcal{T}\bigg(\kappa^2 h^{\mu\lambda} h^{\nu}_{\lambda} \partial_\mu \Phi^\dagger \partial_\nu \Phi - \frac{1}{4}(h_{\lambda\rho})^2 (\partial^\mu \Phi^\dagger \partial_\mu \Phi + M^2 \Phi^\dagger \Phi)\bigg) \middle| i \right\rangle, \tag{3.6}$$

where $|i\rangle = |g, \Phi\rangle\,, |f\rangle = |g, \Phi\rangle$ and we also have

$$\hat{\Phi} \to \frac{\hat{a}_{p_i} e^{ip_i \cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}, \qquad \hat{h}_{\mu\nu} \to \frac{\hat{a}_{q_i\lambda} \epsilon^{\lambda}_{\mu\nu} e^{iq_i \cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}},$$

and

$$\hat{\Phi}^\dagger \to \frac{\hat{a}^\dagger_{p_f} e^{-ip_f \cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}, \qquad \hat{h}_{\mu\nu} \to \frac{\hat{a}^\dagger_{q_f\rho} \epsilon^{\rho}_{\mu\nu} e^{-iq_f \cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}.$$

Let's call the first term S^I_{fi} and the second term S^{II}_{fi}. Then

$$S^I_{fi} = i\kappa^2 \int \mathrm{d}^4x \left\langle f \middle| \mathcal{T}\left(\frac{(-ip_{f\mu})\hat{a}^\dagger_{p_f} e^{-ip_f \cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}\right)\left(\frac{\hat{a}^\dagger_{q_f\alpha} \epsilon^{\mu\lambda}_{\alpha} e^{-iq_f \cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}\right)\left((ip_{i\nu})\frac{\hat{a}_{p_i} e^{ip_i \cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}\right)\left(\frac{\hat{a}_{q_i\beta} \epsilon^{\beta}_{\lambda\nu} e^{iq_i \cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}}\right) \middle| i \right\rangle \tag{3.7}$$

$$= i\kappa^2 \int \mathrm{d}^4x \, \frac{(-ip_f \cdot \epsilon_f)(\epsilon_f \cdot \epsilon_i)(ip_i \cdot \epsilon_i) e^{i(p_i + q_i - p_f - q_f)\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i} 2\mathcal{V}E_{\mathbf{p}_f} 2\mathcal{V}\omega_{\mathbf{q}_i} 2\mathcal{V}\omega_{\mathbf{q}_f}}}, \tag{3.8}$$

and integrating the above, the associated amplitude is

$$i\mathcal{M}^I_{fi} = -\kappa^2 (\epsilon_f \cdot \epsilon_i)(p_f \cdot \epsilon_f)(p_i \cdot \epsilon_i), \tag{3.9}$$

where we used the fact that $\epsilon_{\mu\nu} = \epsilon_\mu \epsilon_\nu$. There's an additional amplitude where we swap the polarization vector and momentum of the graviton

$$i\mathcal{M}^{III}_{fi} = -\kappa^2 (\epsilon_f \cdot \epsilon_i)(p_f \cdot \epsilon_i)(p_i \cdot \epsilon_f). \tag{3.10}$$

Next we set our attention to the other S-matrix term

$$S_{fi}^{II} = -\frac{i\kappa^2}{4}\int \mathrm{d}^4x \,\langle f\big|\mathcal{T}(h^{\mu\nu}h_{\mu\nu}(\partial^\lambda\Phi^\dagger\partial_\lambda\Phi + M^2\Phi^\dagger\Phi))\big|i\rangle \tag{3.11}$$

$$= -i\kappa^2\int \mathrm{d}^4x \left(\frac{\epsilon_\rho^{\mu\nu}e^{-iq_f\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}}\right)\left(\frac{\epsilon_{\mu\nu}^{\lambda}e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}}\right)\left(\frac{e^{-ip_f\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}\right)\left(\frac{e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}\right)\big[(-ip_f^\lambda)(ip_{i\lambda}) + M^2\big] \tag{3.12}$$

$$= -i\kappa^2\int \mathrm{d}^4x \,\frac{(\epsilon_f\cdot\epsilon_i)^2(p_f\cdot p_i + M^2)e^{i(p_i+q_i-p_f-q_f)\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}2\mathcal{V}E_{\mathbf{p}_f}2\mathcal{V}\omega_{\mathbf{q}_i}2\mathcal{V}\omega_{\mathbf{q}_f}}}, \tag{3.13}$$

and when integrating the result gives

$$i\mathcal{M}_{fi}^{II} = \frac{\kappa^2}{4}(\epsilon_f\cdot\epsilon_i)^2(p_f\cdot p_i + M^2) = \frac{\kappa^2}{4}(\epsilon_f\cdot\epsilon_i)^2 q_f\cdot q_i, \tag{3.14}$$

where we used the fact that

$$p_i + q_i = p_f + q_f \Leftrightarrow (p_i - p_f)^2 = (q_i - q_f)^2 \Rightarrow p_f\cdot p_i = q_f\cdot q_i - M^2. \tag{3.15}$$

There is also the other diagram to take account of where we swap the momentum and polarization vector but that diagram is identical to the previous one. Therefore the tree level amplitude for the 4-point vertex is

$$i\mathcal{M}_{fi}^{I}+i\mathcal{M}_{fi}^{II}+i\mathcal{M}_{fi}^{III}+i\mathcal{M}_{fi}^{IV} = -\kappa^2\left[(\epsilon_f\cdot\epsilon_i)(p_f\cdot\epsilon_f p_i\cdot\epsilon_i + p_f\cdot\epsilon_i p_i\cdot\epsilon_f) - \frac{1}{2}(\epsilon_f\cdot\epsilon_i)^2 q_f\cdot q_i\right]. \tag{3.16}$$

Now we can look at the 3-point vertex:

$$S_{fi} = \frac{(-i)^2\kappa^2}{2!}\int \mathrm{d}^4x\int \mathrm{d}^4y \,\langle f|\mathcal{T}(h^{\lambda\rho}(y)h^{\mu\nu}(x)\partial_\lambda\Phi^\dagger(x)\partial_\rho\Phi(x)\partial_\mu\Phi^\dagger(x)\partial_\nu\Phi(x))|i\rangle\,, \tag{3.17}$$

where $|f\rangle\,, |i\rangle$ are the as defined previously. Using

$$\hat{\Phi} \to \frac{\hat{a}_{p_i}e^{ip_i\cdot x}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}}}, \qquad \hat{h}_{\mu\nu} \to \frac{\hat{a}_{q_i\lambda}\epsilon_{\mu\nu}^{\lambda}e^{iq_i\cdot x}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_i}}},$$

and

$$\hat{\Phi}^\dagger \to \frac{\hat{a}_{p_f}^\dagger e^{-ip_f\cdot y}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_f}}}, \qquad \hat{h}_{\mu\nu} \to \frac{\hat{a}_{q_f\rho}^\dagger\epsilon_{\mu\nu}^{\rho}e^{-iq_f\cdot y}}{\sqrt{2\mathcal{V}\omega_{\mathbf{q}_f}}},$$

we are left with

$$S^V_{fi} = -\kappa^2 \int \mathrm{d}^4x \int \mathrm{d}^4y \, \frac{\epsilon_f^{\alpha\beta}\epsilon_i^{\mu\nu}(-ip_{f\alpha})(ip_{i\nu})e^{i(p_i+q_i)\cdot x}e^{-i(p_f+q_f)\cdot y}}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}2\mathcal{V}E_{\mathbf{p}_f}2\mathcal{V}\omega_{\mathbf{q}_i}2\mathcal{V}\omega_{\mathbf{q}_f}}} \left\langle 0\middle|\mathcal{T}(\partial_\mu\Phi^\dagger(x)\partial_\beta\Phi(y))\middle|0\right\rangle . \tag{3.18}$$

Recall from Scalar QED the Green's function

$$\left\langle 0\middle|\mathcal{T}(\partial_\mu\Phi(x)\partial_\beta\Phi^\dagger(y))\middle|0\right\rangle = i\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu k_\beta e^{ik\cdot(x+y)}}{-k^2-M^2+i\varepsilon}, \tag{3.19}$$

the S-matrix is then

$$S^V_{fi} = \frac{i\kappa^2(\epsilon_f\cdot p_f)(\epsilon_i\cdot p_i)\epsilon_i^\mu\epsilon_f^\beta}{\sqrt{2\mathcal{V}E_{\mathbf{p}_i}2\mathcal{V}E_{\mathbf{p}_f}2\mathcal{V}\omega_{\mathbf{q}_i}2\mathcal{V}\omega_{\mathbf{q}_f}}}\int \mathrm{d}^4x\int \mathrm{d}^4y\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{k_\mu k_\beta e^{i(p_i+q_i-k)\cdot x}e^{i(k-p_f-q_f)\cdot y}}{-k^2-M^2+i\varepsilon}. \tag{3.20}$$

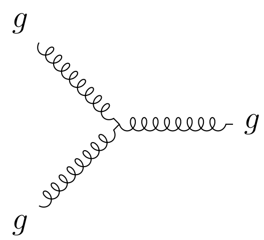
Integrating the x and y coordinates yield $\int \mathrm{d}^4x \to (2\pi)^4\delta^{(4)}(p_i+q_i-k)$, $\int \mathrm{d}^4y \to (2\pi)^4\delta^{(4)}(k-p_f-q_f)$ and integrating the internal momentum yields $\int \mathrm{d}^4k \to \delta^{(4)}(p_i + q_i - p_f - q_f)$. We are thus left with the amplitude

$$i\mathcal{M}^V_{fi} = -\kappa^2\frac{(p_f\cdot\epsilon_f)(p_i\cdot\epsilon_i)\epsilon_i^\mu(p_f+q_f)_\mu\epsilon_f^\beta(p_f+q_f)_\beta}{-(p_f+q_f)^2-M^2} = \frac{\kappa^2}{2}\frac{(p_i\cdot\epsilon_i)^2(p_f\cdot\epsilon_f)^2}{p_i\cdot q_i}. \tag{3.21}$$

And the other amplitude where we swap the q's and ϵ's

$$i\mathcal{M}^{VI}_{fi} = -\frac{\kappa^2}{2}\frac{(p_i\cdot\epsilon_f)^2(p_f\cdot\epsilon_i)^2}{p_i\cdot q_f}. \tag{3.22}$$

Unfortunately, we're not done. There is yet one more diagram of order κ^2 coming from the Einstein-Hilbert action whose interaction vertex is



The cubic order Ricci scalar is

$$\sqrt{-g}R^{(3)} = -\frac{3\kappa^3}{2}h^{\mu\nu}h^{\lambda\rho}\partial_\lambda\partial_\rho h_{\mu\nu} - 9\kappa^3h^{\mu\nu}\partial^\rho h^\lambda{}_\nu\partial_\rho h_{\mu\lambda} - 3\kappa^3\partial^\lambda h^{\mu\nu}\partial^\rho h_{\mu\lambda}h_{\nu\rho}. \tag{3.23}$$