Brans-Dicke Theory

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1 Equations of Motion

Here we are interested in deriving the Brans-Dicke equations of motion. First we start off with the Lagrangian for Brans-Dicke

$$\mathcal{L} = \sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi + \mathcal{L}_m\right). \tag{1}$$

where ϕ is a scalar field, $R = R^\mu{}_\mu = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, ω is a dimensionless parameter and $\mathcal{L}_m$ is the Lagrangian for matter. Next we place the Lagrangian in the action.

$$S = \int \sqrt{-g}(\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi + \mathcal{L}_m)\ \mathrm{d}^4x. \tag{2}$$

We will soon vary with respect to the inverse metric. But first we recognize that $R = g^{\mu\nu}R_{\mu\nu}$. So we then have

$$S = \int \sqrt{-g}\left(\phi g^{\mu\nu}R_{\mu\nu} - \frac{\omega}{\phi}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \mathcal{L}_m\right)\mathrm{d}^4x\,. \tag{3}$$

Now we shall vary the action with respect to the inverse metric

$$\delta S = \delta S_{\phi R} + \delta S_\phi + \delta S_M \tag{4}$$

where

$$\delta S_{\phi R} = \int (\phi R_{\mu\nu}\delta g^{\mu\nu} + \phi g^{\mu\nu}\delta R_{\mu\nu})\sqrt{-g} + \phi R\delta\sqrt{-g}\mathrm{d}^4x,$$
$$\delta S_{\phi} = \int -\frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi\,\delta g^{\mu\nu}\,\sqrt{-g} - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi\,\delta\sqrt{-g}\,\mathrm{d}^4x \tag{5}$$

and S_M is the action for matter. The second term in $S_{\phi R}$ can be found in Carroll's book. Using the result from there we find $\delta S_{\phi R}$ takes the form

$$\begin{aligned}\delta S_{\phi R} &= \int (\phi R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\rho[g_{\mu\nu}\nabla^\rho\phi\delta g^{\mu\nu} - \nabla_\lambda\phi\delta g^{\rho\lambda}]\sqrt{-g} - \frac{1}{2}\phi R\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu})\ \mathrm{d}^4x \\ &= \int (\phi R_{\mu\nu} - \frac{1}{2}\phi R g_{\mu\nu} - [\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi])\sqrt{-g}\delta g^{\mu\nu}\ \mathrm{d}^4x.\end{aligned} \tag{6}$$

Looking now to δS_ϕ, the action becomes

$$\delta S_\phi = -\int \frac{\omega}{\phi}(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2)\sqrt{-g}\delta g^{\mu\nu}\ \mathrm{d}^4x, \tag{7}$$

with $\nabla_\mu\phi\nabla^\mu\phi = (\nabla\phi)^2$. Recall that the functional derivative of the action satisfies

$$\delta S = \int \sum_i \left(\frac{\delta S}{\delta\Psi^i}\,\delta\Psi^i\right)\mathrm{d}^dx\,, \tag{8}$$

where $\{\Psi^i\}$ is a complete set of fields being varied. This brings the total action δS to be

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \phi\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi\right) - \frac{\omega}{\phi}(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu_\nu}(\nabla\phi)^2) + \frac{1}{2\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \tag{9}$$

Defining the energy momentum tensor to be

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}} \tag{10}$$

Moving the last terms to the other side and dividing both sides by ϕ, we get

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2\phi}T_{\mu\nu} + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi) + \frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right) \tag{11}$$

2 Degrees of Freedom

Brans-Dicke theory was one of the earliest competitors to GR. It is discussed today because it represent the prototypical example of a scalar tensor theory. The Lagrangian for the theory is given by

$$\mathcal{L} = \sqrt{-g}\bigg(\phi R - \frac{\omega}{\phi}\nabla_\mu \phi \nabla^\mu \phi\bigg). \tag{12}$$

We wish to expand all the terms to at most second order in ϕ, R, and $\sqrt{-g}$. This brings us to

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{13}$$

$$\phi = \phi_o + \varphi, \tag{14}$$

$$\sqrt{-g} = 1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu}, \tag{15}$$

$$R = R^{(0)} + R^{(1)} + R^{(2)}, \tag{16}$$

where η and ϕ_o is some flat space solution, $h_{\mu\nu}$ and φ are perturbations and taken to be much less than 1, $h = h^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu}$ is the trace , and $R^{(0)} = 0,\ R^{(1)},\ R^{(2)}$ are the zeroth, first, and second order curvature respectively. Keeping only the terms quadratic in both $h_{\mu\nu}$ and φ, the Lagrangian becomes

$$\mathcal{L} = \phi_o(\sqrt{-g}R)^{(2)} - \frac{\omega}{\phi_o}\partial_\mu\varphi\partial^\mu\varphi. \tag{17}$$

Lets split these Lagrangians off into different sectors in order to deal with them easily.

$$\mathcal{L}_{R^{(2)}} = \phi_o(\sqrt{-g}R)^{(2)}, \tag{18}$$

$$\mathcal{L}_\varphi = -\frac{\omega}{\phi_o}\partial_\mu\varphi\partial^\mu\varphi, \tag{19}$$

Since we've done the degree of freedom count for $\mathcal{L}_{R^{(2)}}$ (being multiplied by a constant background won't change any of the calculations) we only need to focus our attention on $\mathcal{L}_{\varphi}$. We next perform the following decompositions:

$$h_{00} = h^{00} = 2\Phi, \tag{20}$$

$$h_{0i} = -h^0_i = w_i, \tag{21}$$

$$h = h^{\mu}{}_{\mu} = \eta^{\mu\nu} h_{\mu\nu} = -2\Phi + \bar{h}, \tag{22}$$

$$w_i = w_i^T + \partial_i \Lambda, \tag{23}$$

$$h_{ij} = h_{ij}^{TT} + \partial_i v_j^T + \partial_j v_i^T + 2\left(\partial_i \partial_j \Psi - \frac{1}{3}\nabla^2 \Psi \delta_{ij}\right) + \frac{1}{3}\bar{h}\delta_{ij}, \tag{24}$$

$$\partial^i h_{ij}^{TT} = \delta^{ij} h_{ij}^{TT} = \partial^i v_i^T = \partial^i w_i^T = 0, \tag{25}$$

where $\bar{h} = Tr[h_{ij}]$ and δ_{ij} is the identity matrix. Defining the following gauge invariant fields

$$J \equiv \Phi - \dot{\Lambda} + \ddot{\Psi}, \qquad L \equiv \frac{1}{3}(\bar{h} - 2\nabla^2 \Psi), \qquad M_i \equiv w_i^T - \dot{v}_i^T. \tag{26}$$

We're ready to start our analysis on the degrees of freedom in these actions. We can turn our gaze to $\mathcal{L}_{R^{(2)}}$ and using the results from the linearized GR calculation while also scaling by the constant background ϕ_o we get the following actions:

$$S_T = \int \frac{1}{2} h^{ij}_{TT} \Box h_{ij}^{TT} \; \mathrm{d}^4 x, \tag{27}$$

$$S_V = \int \frac{1}{2} (\partial_i M_j)^2 \; \mathrm{d}^4 x, \tag{28}$$

$$S_S = \int 4J\nabla^2 L - L\nabla^2 L - 2\dot{L}^2 \; \mathrm{d}^4 x, \tag{29}$$

$$S_\varphi = \int \frac{\omega}{\phi_o}\varphi\Box\varphi \; \mathrm{d}^4x. \tag{30}$$

We can now analyze the true degrees of freedom that are present in $h_{\mu\nu}$. First, looking at the vector action we can see that no time derivatives of M_i are present in the action. Therefore it is an auxiliary field and we may use its equations of motion (EOM) to eliminate it. Proceeding accordingly we find

$$\frac{\delta\mathcal{L}}{\delta M^i} = \nabla^2 M_i = 0 \Rightarrow M_i = 0, \tag{31}$$

which implies that $S_V = 0$. Next we turn our attention to the scalar action. Since J appears linearly with no time derivatives, we may interpret it as a Lagrange multiplier. From there we can see that the EOM of J enforces the following constraint:

$$\frac{\delta\mathcal{L}}{\delta J} = \nabla^2 L = 0 \Rightarrow L = 0, \tag{32}$$

and therefore, $S_S = 0$. The total action is now $S = S_T + S_\varphi$ and thus we can say that the Lagrangian for linearized Brans-Dicke has in total, three degrees of freedom. two from h_{ij}^{TT} and one from φ.