An Overview on Modified Theories of Gravity

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1 Preface

We are interested in exploring the richness of theories of modified gravity (MG). Before we really dive in, it is necessary to answer the question on what exactly do we mean by modified theories of gravity. Broadly speaking, modified gravity refers to two things: theories whose equations of motion admits the Einstein Field Equations (EFE) and Modified Newtonian Dynamics (MOND). We shall be spending the majority of this document discussing the former, but we shall briefly go into the latter as well. We will see that there have been many theories that have been cooked up over the decade with each new approach to modifying General Relativity in such a way where (1) it can replicate GR in all regimes where we know GR is correct/an accurate description and (2) makes predictions distinct from GR in regions that have yet to be probed.

Conventions We use the mostly plus metric signature, i.e. $\eta_{\mu\nu} = (-,+,+,+)$ and units where $c = \hbar = 1$. The reduced four dimensional Planck mass is $M_P = \frac{1}{\sqrt{8\pi G}} \approx 2.43 \times 10^{18}$ GeV. The d'Alembert and Laplace operators are defined to be $\Box = \partial_\mu \partial^\mu$ and $\nabla^2 = \partial_i \partial^i$ respectively. We use boldface letters $\mathbf{x}$ to indicate 3-vectors and we use x and p to denote 4-vectors. Conventions for the curvature tensors, covariant and Lie derivatives are all taken from Carroll [1].

2 Introduction

General Relativity (GR) has proven itself time and time again as one of our most successful theories in all of physics. From the bending of light, to the prediction of Mercury's precession around the sun, to the Shapiro time delay, GR has passed every classical (solar system) test that has been thrown at it and serves as the backbone to the standard model of cosmology. To this day, the usage of GPS systems all around the world lend their existence due to this beautiful theory. And yet there are still a number of theorists who seek to modify it in some way.

3 Motivation

Historically, the motivation for studying MG was to better understand GR. Being able to alter and deform GR can give a substantial insight into the sensitivity of its mathematical structure. We find that there can only be very small deviations or very tight constraints that are placed on any new parameters to the theory.

In the modern day, when asked for their motivation, theorists cite the expansion of the universe. Data gathered from Type 1a supernovae leads one to the conclusion that the universe's rate of expansion is increasing which is to say that it is accelerating exponentially. If GR is to be believed (and we have every reason to do so) then this accelerated expansion is due to the so-called Dark Energy. Reports of the rotation curves of galaxies also paint the picture that our understanding of the universe on an astronomical scale is also lacking.

What's more, all attempts on building a full quantum field theory (QFT) out of GR using our usual tools have all proved to be unsuccessful so far. GR is non-renormalizable as a QFT i.e. it is not UV complete. With the Planck scale being at $M_P = 2.43 \times 10^{18}$ GeV in addition to any and all phenomenon being well below this energy threshold, we can well regard GR as an effective field theory. These observations combined leads us to the following conclusion: if GR represents our best understanding of gravity then our understanding is yet to be complete.

4 Lovelock's Theorem

So what kinds of theories are we even allowed to write down?

Theorem 4.1 (Lovelock's Theorem) *Suppose $A^{\mu\nu}$ is a symmetric, bilinear, divergence-less tensor on a four dimensional (pseudo-)Riemannian manifold and that is a function of the metric and up to its second derivatives (although linear in its second derivatives). Then, the only possible tensor is*

$$A^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu}, \tag{1}$$

where $G^{\mu\nu}$ is the Einstein tensor and a, b are constants.

This is a very powerful theorem because it places very heavy constraints on what we are allowed to do the original Einstein-Hilbert action and expect to still get something proportional to the EFE. This implies that the only modifications to GR that are allowed has to fall into one of the five categories:

- Add additional degrees of freedom to the action
- Work with derivatives of the metric higher than four
- Work in $d \neq 4$ dimensional spacetime
- Introduce non-local interactions into the Lagrangian i.e. inverse differential operators/Green's functions to the action
- Deduce the equations of motion with something other than the Euler-Lagrange equations

With all of the restrictions that are placed on the equations of motion, one might wonder whether its possible to add only tensors whose components are functions of the second derivatives of the metric raised to exactly one power. This leads us to our second most powerful theorem

Theorem 4.2 *Let R be the curvature scalar of some (pseudo-)Riemannian manifold. Then, we can say that it is the only scalar invariant which is linear in the second derivatives of the associated metric tensor $g_{\mu\nu}$.*

Since we're required to only include objects that are invariant under both local Lorentz transformation as well as arbitrary diffeomorphisms, the above theorem tells us that the only allowed scalar quantity (and hence the only allowed object in the Lagrangian) is precisely the Ricci scalar. This theorem is usually referred to as Vermeil's theorem, after the German mathematician who proved it in 1917.

Now that we have laid down the groundwork, we are finally ready to be introduced to a survey of the possible models that exist in the literature.

5 Te(Ve)S

The first class of modified gravity theories we want to cover is what happens when one includes additional degrees of freedom to the action. This class of theories goes by the name Tensor-(Vector)-Scalar theories or Te(Ve)S for short. These are probably the most popular method of modifying GR because of how flexible theories of this kind can be. Since vector degrees of freedom are often left out, we shall do the same here as well.

5.1 Brans-Dicke Theory

Brans-Dicke Theory is sort of the prototypical example of a scalar field theory. It was considered to be competitor of GR in the earlier days of the theory, but observations have placed very heavy constraints on the parameters of this scalar-tensor theory so as to render it as a toy model for modified gravity. The Lagrangian for Brans-Dicke theory is

$$\mathcal{L} = \sqrt{-g}\left(\phi R - \frac{\omega}{\phi}\nabla_\mu \phi \nabla^\mu \phi + \mathcal{L}_m\right), \tag{2}$$

where ϕ is a scalar field, ω is a dimensionless parameter and $\mathcal{L}_m$ is the Lagrangian for matter. The equations of motion for the metric are then

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2\phi}T_{\mu\nu} + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi) + \frac{\omega}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla\phi)^2\right), \tag{3}$$

where $(\nabla\phi)^2 \equiv \nabla_\mu\phi\nabla^\mu\phi$ and we have defined the stress-energy tensor to be

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{4}$$

with S_M being the action for the matter fields. Next we write down the equations of motion for the scalar field

$$\Box\phi = \frac{1}{2(2\omega+3)}T. \tag{5}$$

Current constraints on the dimensionless parameter ω give us $|\omega| \gtrsim 10^4$.

5.2 Horndeski Theory

Horndeski Theory represents *the most general scalar-tensor Lagrangian in four dimensions that admits at most second derivatives of the scalar field in its equations of motion.* This is a requirement that's imposed on us by Ostrogradsky's Instability to avoid ghost instabilities in our equations of motion. The most general Lagrangian that we can write down is given by

$$\begin{aligned}\mathcal{L}_H = {} & G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4,X}\left[(\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right] \\ & + G_5(\phi, X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{G_{5,X}}{6}\left[(\Box\phi)^3 - 3\Box\phi\nabla^\mu\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi + 2\nabla_\mu\nabla_\nu\phi\nabla^\nu\nabla^\lambda\phi\nabla_\lambda\nabla^\mu\phi\right]\end{aligned} \tag{6}$$

where $X = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, $G_i = G_i(\phi, X)$ are arbitrary functions of ϕ and X, and we use $f_{,X}$ and $f_{,\phi}$ to denote derivatives with respect to X and ϕ respectively. The theory as written is actually called the generalized Galileon but its closely related to Horndeski Theory. This theory is important because all scalar-tensor theories can be cast as a limiting case of the above theory just by making particular choices of G_i's. For example, notice how we recover Brans-Dicke theory by simply setting $G_2(\phi, X) = \frac{2\omega}{\phi}X$, $G_4(\phi, X) = \phi$, and $G_3 = G_5 = 0$.

5.3 Massive Gravity

Typically in the context of GR, one includes the mass term by way of expanding the metric around some background spacetime i.e.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{7}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $|h_{\mu\nu}| << 1$ is the metric perturbation which will serve as our dynamical field. We work on flat spacetime WLOG. The mass term is simply

$$\mathcal{L}_{m\neq 0} = -\frac{1}{8\kappa^2}m^2(h^{\mu\nu}h_{\mu\nu} - h^2) = -\frac{m^2}{8\kappa^2}\eta^{\mu\lambda}\eta^{\nu\rho}(h_{\mu\nu}h_{\lambda\rho} - h_{\mu\lambda}h_{\nu\rho}), \tag{8}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$ is the trace of the perturbation and $\kappa = \sqrt{8\pi G}$ is the reduced Planck mass. Notice how the addition of this new mass term includes all possible quadratic contractions of h as is the case in E&M when one adds the massive vector $\frac{1}{2}m^2 A_\mu A^\mu$. The mass term gets added to the quadratic curvature scalar ala

$$\mathcal{L} = R^{(2)} - \frac{1}{2}m^2(h^{\mu\nu}h_{\mu\nu} - h^2) \tag{9}$$

$$= \partial_\lambda h_{\mu\nu}\partial^\mu h^{\lambda\nu} + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - \partial_\mu h^{\mu\nu}\partial_\nu h - \frac{1}{2}m^2(h^{\mu\nu}h_{\mu\nu} - h^2) \tag{10}$$

There's a number of interesting complications that arise when considering a massive spin-2 field. The first issue is the loss of gauge symmetry as is the case when one introduces a massive vector field in the case of the Proca Lagrangian. This deficiency can be ameliorated by performing the St*ü*kelberg Trick in that one adds redundant fields to the action that restores the gauge symmetry. Secondly, we don't smoothly recover GR in the limit as $m^2 \to 0$. Massive gravity carries 5 degrees of freedom while GR only carries 2 degrees of freedom which correspond to the two polarizations of the graviton. Even if we didn't have the aforementioned issues associated to this theory, there is still the issue of ghost instabilities. There have been constructions of massive gravity that have rendered this a non-issue however.

6 f(R) Theory

Here we write about what happens when one wishes to consider derivatives that are higher than two in the metric. This class of theories are broadly referred to as $f(R)$ models, where R is the Ricci scalar. Now theories that admit higher order derivatives greater than two in the dynamical field are generally disfavored due to Ostrogradsky Instability. However, $f(R)$ theories admit a particular symmetry which may render this a non-issue.

The action for $f(R)$ theories is given by the following

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}f(R), \tag{11}$$

where again $f(R)$ is some general function of the curvature scalar. When we vary the action with respect to the metric, we get

$$f'(R)R_{\mu\nu} - (\nabla_\mu\nabla_\nu f'(R) - \Box f'(R)g_{\mu\nu}) - \frac{1}{2}f(R)g_{\mu\nu} = 0, \tag{12}$$

where primes denote derivatives with respect to the curvature scalar. It is also common in the literature to write

$$f_R \equiv \frac{\mathrm{d}f}{\mathrm{d}R}, \tag{13}$$

and we call f_R the scalaron. Note, we recover the EFE by setting $f(R) = R$. Without loss of generality, we can also include the Gauss-Bonnet term in the action

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}[f(R) + \mathcal{G}], \quad \mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}. \tag{14}$$

The Gauss-Bonnet term is a topological term that exists for (pseudo-)Riemannian manifolds in $d = 4$ dimensions. It's typically left out of the action because it is a total derivative term and hence plays no role in the dynamics of our theories.

6.1 Hu-Sawicki Theory

The model posits that the action should be

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R + f(R)) + S_m \tag{15}$$

where

$$f(R) = -m^2\frac{c_1(R/m^2)^n}{c_2(R/m^2)^n + 1}, \tag{16}$$

S_m is the action for any additional matter fields. We have three free parameters c_1, c_2, and n, and the parameter $m^2 = \kappa\bar{\rho}_m/3$ with $\bar{\rho}_m$ being the average matter density of the universe. For large curvature such that $|m^2/R| \ll 1$, the scalaron becomes

$$f(R) \approx -\frac{c_1}{c_2}m^2 + \frac{c_1}{c_2^2}m^2\left(\frac{m^2}{R}\right)^n. \tag{17}$$

This implies as $c_1/c_2^2 \to 0$, we recover ΛCDM cosmology i.e. c_1/c_2 approaches the cosmological constant. Now, in order to replicate the entire expansion history of the universe, the curvature scalar has to take on a background value of

$$\overline{R} = 3\Omega_{m,0}H_0^2\left(1 + 4\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right), \tag{18}$$

where $\Omega_{\Lambda,0}$ is the dark energy fractional energy density evaluated in the present time. The additional scalar degree of freedom (also known as the scalaron) plays a very important role in the growth an evolution of structure. As a result, we're interested in studying its behavior. Since its first derivative governs its evolution we write

$$f_R = -n\frac{c_1(R/m^2)^{n-1}}{(c_2(R/m^2)^n + 1)^2} \approx -n\frac{c_1}{c_2^2}\left(\frac{m^2}{R}\right)^{n+1}. \tag{19}$$

Plugging in the background value for the curvature scalar gives us

$$\bar{f}_{R_0} = -n\frac{c_1}{c_2^2}\left(\frac{\Omega_{m,0}}{3(\Omega_{m,0} + \Omega_{\Lambda,0})}\right)^{n+1}. \tag{20}$$

6.2 TeVeS Conformal Equivalence

What makes $f(R)$ models special, is the fact that they are conformally equivalent to scalar-tensor theories. And this equivalence is most apparent under a Legendre transformation. First we write

$$f(R) = R + \tilde{f}(R) \Rightarrow \phi(R) \equiv \frac{\mathrm{d}f}{\mathrm{d}R} = 1 + \tilde{f}_R, \tag{21}$$

where ϕ acts as the canonical partner to $f(R)$. If we assume that $\phi(R)$ is an invertible function i.e. $R(\phi)$ exists, then we can define a potential for our scalar field and our Lagrangian becomes

$$\mathcal{L}_E = \sqrt{-g}(\phi R - U(\phi)), \tag{22}$$

where the potential $U(\phi)$ is given by

$$U(\phi) = (\phi - 1)R - \tilde{f}(R). \tag{23}$$

Now our Lagrangian is currently unwieldy. It is expressed what theorists call the *Jordan Frame*, or the scalar field is non-minimally coupled to gravity. We can change to the *Einstein Frame* by a simple conformal transformation on the metric i.e.

$$\bar{g}_{\mu\nu} = \phi g_{\mu\nu}. \tag{24}$$

This turns the Lagrangian into the form

$$\mathcal{L}_E = \sqrt{-g}\left(\frac{1}{2\kappa^2}\bar{R} - \frac{1}{2}\bar{g}^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right), \tag{25}$$

where we have introduced the new scalar field φ and its associated potential via the relations

$$\phi = \exp\left(\sqrt{\frac{\kappa^2}{6}}\varphi\right), \quad V(\varphi) = \frac{1}{2\kappa^2}\exp\left(-\sqrt{\frac{2\kappa^2}{3}}\varphi\right)U(\varphi). \tag{26}$$

And thus we are able to escape many of the same pathologies that are brought up when considering higher order derivatives of the metric by simply swapping for some generic scalar field.

7 d > 4 Dimensions

This class of theories broadly refers to working in spacetime dimensions d, such that $d \neq 4$. Since gravity is trivial in $d < 4$ dimensions, this requirement almost always ends up referring to $d > 4$ dimensions.

7.1 Kaluza-Klein Theory

Kaluza-Klein theory is thought to be one of the very first attempts that ended up unifying the fundamental forces to another. Kaluza-Klein theory is important for a number of historical reasons. It is the theory that first bore the idea that there could be extra spatial dimensions that are yet to be accounted for in our Standard Model of particle physics. It is also the theory that gave birth to the idea that we can curl the additional spatial dimension i.e. compactify it so as to hide any detection of that new spatial dimension. These ideas have clearly found themselves into the mainstream consciousness in the research program of String Theory. We start off with the line element

$$\mathrm{d}s^2 = g_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B = e^{2\alpha\phi}g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + e^{2\beta\phi}(\mathrm{d}x^4 + A_\mu\,\mathrm{d}x^\mu)^2, \tag{27}$$

where capital Latin indices run from 0 to 4 and lowercase indices run from 0 to 3, ϕ is a scalar field often called the dilaton and A_μ is some vector field and α, β are constants defined as

$$\alpha = \frac{1}{2\sqrt{(d-1)(d-2)}}, \qquad \beta = -(d-2)\alpha, \tag{28}$$

where $d = 4$ is the number of dimensions after integrating out the extra dimension. The action for this theory becomes

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left(R - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{4}e^{-\sqrt{3}\phi}F^{\mu\nu}F_{\mu\nu}\right), \tag{29}$$

where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. The equations of motion found from varying $g_{\mu\nu}$, ϕ, and A_μ respectively are

$$G_{\mu\nu} = \frac{1}{2}\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\lambda\phi\nabla^\lambda\phi + e^{-\sqrt{3}\phi}\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{2}F^2 g_{\mu\nu}\right)\right], \tag{30}$$

$$\nabla^\mu(e^{-\sqrt{3}\phi}F_{\mu\nu}) = 0, \tag{31}$$

$$\Box\phi = -\frac{\sqrt{3}}{4}e^{-\sqrt{3}\phi}F^2, \tag{32}$$

where $F^2 = F^{\mu\nu}F_{\mu\nu}$. Notice that a vanishing dilaton field enforces the condition that $F^2 = 0$ which implies no new dynamics.

7.2 DGP Gravity

The DGP model posits that our four dimensional spacetime is embedded within a five dimensional brane. The action for the theory is

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{2\kappa^2}R + \mathcal{L}_m\right) + \frac{1}{2\kappa^2 r_c}\int \mathrm{d}^5x\,\sqrt{-g_5}R_5, \tag{33}$$

where r_c is the free parameter that determines the length-scale for which we recover standard ΛCDM cosmology, and R_5 and g_5 are the five dimensional curvature scalar and metric determinant respectively. The equations of motion are

$$\nabla^2\phi = 4\pi G a^2\delta\rho_m + \frac{1}{2}\nabla^2\varphi, \tag{34}$$

$$\nabla^2\varphi + \frac{r_c^2}{3\beta a^2}\left[(\nabla^2\varphi)^2 - (\nabla_i\nabla_j\varphi)^2\right] = \frac{8\pi G a^2}{3\beta}\delta\rho_m, \tag{35}$$

where

$$\beta = \beta(a) = 1 + 2Hr_c\left(1 + \frac{\dot{H}}{3H^2}\right). \tag{36}$$

8 Non-Locality

A non-local action generally refers to an action that possesses functions with non-polynomial differential operators. These can be broken down into two classes: analytic and non-analytic. Analytic differential operators take on the form

$$\mathcal{L} = R + R\left(\frac{e^{-\Box/\ell^2} - 1}{\Box}\right)R. \tag{37}$$

Non analytic functions typically are represented by inverse differential operators i.e. Green's functions

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\big[R + Rf(\Box^{-1}R) - 2\Lambda\big]. \tag{38}$$

They receive the label of non-locality because a Green's function $G(x-y)$ is a function of two spacetime events that, in principle, can be taken to be arbitrarily far away from one another

9 MOND

Modified Newtonian Dynamics (MOND) refers to a modification to the gravitational potential in the Poisson equation

$$\nabla^2\Phi = 4\pi G\rho_m. \tag{39}$$

At its heart, MOND makes the radical divergence to modify the Newtonian force law directly instead of the EFE. Milogram proposed the following modification to the Newtonian force law:

$$F = m\frac{a^2}{a_0}, \tag{40}$$

where a_0 is some acceleration scale, produces a flat rotation curve i.e. a constant velocity. Assuming a circular orbit, the velocity is related to the acceleration by

$$a = \frac{v^2}{r}. \tag{41}$$

Since the only force on galactic scales to be considered is the gravitational force, we write

$$\frac{GMm}{r^2} = \frac{m(\frac{v^2}{r})^2}{a_0} \Rightarrow v(r) = (GMa_0)^{\frac{1}{4}}. \tag{42}$$

This all implies the gravitational potential is modified to be

$$-\nabla\Phi = \mu\left(\frac{|\mathbf{a}|}{a_0}\right)\mathbf{a}, \tag{43}$$

where

$$\mu(x) = \begin{cases} x, x \ll 1 \\ \\ 1, x \gg 1 \end{cases} . \tag{44}$$

It has been estimated that $a_0 \approx 1 \times 10^{-8}\,\mathrm{cm\,s^{-2}}$. While MOND is able to explain galactic rotation curves (as well as an impressive array of other astrophysical phenomenon) it is limited in its scope. Because it provides a modification to the gravitational potential in the weak limit, it can't even hope to make predictions on cosmological scales. Never mind respecting all the other symmetries we impose on our gravitational theories such as diffeomorphism invariance and Lorentz invariance. All of that being said, a relativistic extension of MOND was written down by Jacob Bekenstein. By introducing a timelike vector field i.e. A_μ such that

$$g^{\mu\nu}A_\mu A_\nu = -1, \tag{45}$$

as well as a dynamical scalar field ϕ and a non-dynamical scalar field Ω, the action becomes

$$S = S_{EH} + S_V + S_S, \tag{46}$$

where S_{EH} is the regular Einstein-Hilbert action, S_V is the action for the vector field given by

$$S_V = -\frac{K}{4\kappa^2}\int \mathrm{d}^4x\,\sqrt{-g}\left[F_{\mu\nu}F^{\mu\nu} - \frac{2}{K}\lambda(A_\mu A^\mu + 1)\right], \tag{47}$$

where K is a dimensionless parameter, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor for the vector field, and $\lambda = \lambda(x)$ acts as a Lagrange multiplier that enforces the timelike constraint on the vector field. The scalar action is given by

$$S_S = -\frac{1}{2}\int \mathrm{d}^4x \left[\Omega^2 h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\frac{G}{\ell^2}\Omega^4 f\big(kG\Omega^2\big)\right], \tag{48}$$

where $h^{\mu\nu} = g^{\mu\nu} - A^\mu A^\nu$, f is a free dimensionless function, k is a dimensionless constant, and ℓ is introduced for dimension-full consistency. The equations of motion then become

$$G_{\mu\nu} = 8\pi G\big[T_{\mu\nu} + (1-e^{-4\phi})A^\lambda T_{\lambda(\mu}A_{\nu)} + \tau_{\mu\nu}\big] + \Theta_{\mu\nu}, \tag{49}$$

$$K\nabla_\nu F^{\nu\mu} + \lambda A^\mu + 8\pi G\Omega^2 A^\lambda \partial_\lambda\phi g^{\mu\nu}\partial_\nu\phi = 8\pi G(1-e^{-4\phi})g^{\mu\lambda}A^\rho T_{\lambda\rho}, \tag{50}$$

$$\nabla_\nu[\Omega^2 h^{\mu\nu}\partial_\mu\phi] = \big[g^{\mu\nu} + (1+e^{4\phi})A^\mu A^\nu\big]T_{\mu\nu}, \tag{51}$$

$$-kG\Omega^2 f - \frac{1}{2}(kG\Omega)^2 f' = k\ell^2 h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi, \tag{52}$$

where we define the tensors

$$\tau_{\mu\nu} = \Omega^2\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - A^\lambda\partial_\lambda\phi\left(A_{(\mu}\partial_{\nu)}\phi - \frac{1}{2}g_{\mu\nu}A^\rho\partial_\rho\phi\right)\right] - \frac{G}{4\ell^2}\Omega^4 f(kG\Omega^2)g_{\mu\nu}, \tag{53}$$

where $(\partial\phi)^2 \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and

$$\Theta_{\mu\nu} = K\left(F^2_{\mu\nu} - \frac{1}{4}F^2 g_{\mu\nu}\right) - \lambda A_\mu A_\nu, \tag{54}$$

where $F^2_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda$.

10 Constraints Placed on Modified Gravity

Now that we have a nice survey on some of the more popular modified theories of gravity models that are out there in the modern day, we can start talking about different constraints that are imposed on each gravity theory. The constraints on MG generally come from considering cosmological perturbation theory, so we will restrict our attention to this area of physics. When discussing MG theories, it is most convenient to introduce certain parametrizations so that we can avoid having to refer to any particular model and hence we can study entire classes of theories. A popular method for doing this is called the $\mu - \gamma$ parametrization. We start off with the differential line element in the Newtonian gauge under the weak field limit

$$\mathrm{d}s^2 = -[1 + 2\Psi(\mathbf{x}, t)]\,\mathrm{d}t^2 + a^2(t)[1 + 2\Phi(\mathbf{x}, t)]\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,, \tag{55}$$

where $a(t)$ is the scale factor, δ_{ij} is the Kronecker delta, and Ψ and Φ are gravitational potentials. The equations of motion become

$$-k^2\Psi = 4\pi G a^2 \mu(\mathbf{k}, t)\bar{\rho}_m\delta_m, \quad \Phi(\mathbf{k}, t) = \gamma(\mathbf{k}, t)\Psi(\mathbf{k}, t), \tag{56}$$

where δ_m is the overdensity of matter and we express the gravitational potentials in Fourier/momentum space. Note $\mu = \gamma = 1$ recovers GR. In general, for any MG theory to be a true successor to GR, it must pass all of the solar system tests i.e. the precession of Mercury, the Shapiro Time Delay etc. Today, there are additional implications from the discoveries of gravitational waves and the recently imaged "photo" of Sagittarius A*.

10.1 GW170817

GW170817 is the gravitational wave observation from a binary Neutron star inspiral that also had a counterpart in the optical range i.e. a photon. This multi-messenger observation places tight bounds on the speed that gravity has to satisfy according to our MG models. The reason being, the speed of light c and the speed of gravity c_g were found to coincide so closely $|c - c_g| \gtrsim 10^{-15}$, that many models were ruled out by this event.

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